An Axiomatic Model of Persuasion

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Abstract

A sender ranks information structures knowing that a receiver processes the information before choosing an action affecting them both. The sender and receiver may differ in their utility functions and/or prior beliefs, yielding a model of dynamic inconsistency when they represent the same individual at two points in time. I take as primitive (i) a collection of preference orderings over all information structures, indexed by menus of acts (the sender's ex-ante preferences for information), and (ii) a collection of correspondences over menus of acts, indexed by signals (the receiver's signal-contingent choice(s) from menus). I provide axiomatic representation theorems characterizing the sender as a sophisticated planner and the receiver as a Bayesian information processor, and show that all parameters can be uniquely identified from the sender's preferences for information. I also establish a series of results characterizing common priors, common utility functions, and intuitive measures of disagreement for these parameters—all in terms of the sender's preferences for information.

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1 Introduction

This paper develops axiomatic foundations for a general model of communication with sender commitment power. As in Kamenica and Gentzkow (2011), there are two agents: Sender and Receiver. By controlling information, Sender attempts to guide Receiver toward actions that are more beneficial to himself. The main result is a representation theorem characterizing Sender as a sophisticated Bayesian planner and Receiver as a Bayesian information processor. Importantly, Sender's *preference for information* is an observable primitive, reflecting the idea that he controls the information (not the actions) available to Receiver.

Sender and Receiver are expected utility maximizers but may differ in their utility functions or prior beliefs. This enables two interpretations of the model. In the *persuasion* interpretation, Sender and Receiver represent distinct individuals, as in "Bayesian persuasion" models (Kamenica and Gentzkow, 2011). In the *behavioral* interpretation, they represent the same individual at two points in time, yielding a model of dynamically inconsistent behavior. As is well known, sophisticated, dynamically inconsistent individuals value commitment power (Strotz, 1955). Here, Sender lacks hard commitment power in that he cannot restrict the set of actions available to Receiver (his future self). Instead, he selects the information structure that will deliver a signal to Receiver. This provides an alternative form of commitment power, making preferences for information reflect preferences for commitment.

To illustrate the main ideas, as well as the behavioral interpretation, consider an individual who must decide whether to consume a dessert (action D) or not (action $\neg D$). An ingredient in the dessert is either unhealthy (state G) or very unhealthy (state B). In period 1, before the decision is to be made, the individual is health conscious: he prefers not to consume the dessert regardless of the state (preferences v below). He recognizes, however, that he may succumb to temptation when confronted with the choice: his future self prefers to consume the dessert in state G but to refrain in state B (preferences u).[1]

Lacking hard commitment power, the period-1 self (Sender) attempts to influence future choice through careful exposure to information. For example, he may consult a specialist who reveals the true state, or browse web sites containing imperfect information about the state. If he acquires sufficient evidence of state B, his period-2 self (Receiver) will refrain from consuming the dessert despite the lack of hard commitment power.

Differences between first- and second-period utility functions induce non-trivial preferences for information. If, for example, both selves assign prior probability 2/3 to state G, then Sender prefers perfect information over no information: perfect information results in

[1] This payoff structure is isomorphic to that of the leading example in Kamenica and Gentzkow (2011), although the interpretation is different.

	G	B
D	0	-2
$\neg D$	1	1

(a) Utilities v

	G	B
D	2	0
$\neg D$	1	1

(b) Utilities u

choice $\neg D$ with probability 1/3, while no information results in choice D with probability 1. However, perfect information is not ideal from Sender's perspective. Consider the following information structure, denoted σ:

	s	t
G	1/4	3/4
B	1	0

This information structure generates signal s in state B, while in state G it generates s with probability 1/4 and t with probability 3/4. Under Bayesian updating, Receiver chooses D at signal t and $\neg D$ at s. Thus, Sender achieves a higher expected payoff from σ than from perfect information, so that his preference for information violates the Blackwell (1951, 1953) information ordering. Similarly, non-common priors also lead to violations of the Blackwell ordering.[2] A key finding of this paper is that such violations are very informative and that, in fact, Sender's preferences for information fully reveal the priors and utilities of both agents.

In the representation, Receiver selects among *acts* (Anscombe and Aumann, 1963): profiles $f = (f_\omega)_{\omega\in\Omega}$ assigning lotteries $f_\omega \in \Delta X$ to states $\omega \in \Omega$, where X and Ω are finite sets of outcomes and states, respectively. Information structures take the form of Blackwell experiments which, as illustrated above, are matrices σ where each column represents a signal and each row ω represents a state-contingent probability distribution over the signals.

Receiver's choices are summarized by a family of *signal-contingent choice correspondences* c^s. A *signal* is a profile $s = (s_\omega)_{\omega\in\Omega}$ of entries from $[0, 1]$ with at least one non-zero entry. So, s represents a column from some experiment and the entries of s represent likelihoods of the signal being generated in different states of the world. For a signal s and *menu* A (a finite set of acts), $c^s(A) \subseteq A$ is the set of acts chosen by Receiver after observing s. In the representation, choices c^s are rationalized by expected utility maximization with utility index u, prior μ (full support), and Bayesian updating. The key axiom, *Bayesian Consistency*, expresses an equivalence between scaling signal likelihoods and outcome probabilities, ensuring Receiver is a Bayesian updater.

[2]Heterogeneous priors can be interpreted as a different source of temptation. In this example, both decision makers could hold utility function u while the second-period prior is skewed in favor of state G. Thus, the effect of temptation is to become biased or delusional in favor of state G, making the dessert seem more attractive. The decision maker knows himself well enough to anticipate this behavior.

Sender's preferences are summarized by a family of preference relations $\succsim^A$ indexed by menus A. Each $\succsim^A$ is an ordering of the set of all Blackwell experiments and represents Sender's *preference for information* when Receiver must choose from A. The statement $\sigma \succsim^A \sigma'$ means Sender prefers to expose Receiver to information σ over information σ', given that his outcome is determined by Receiver's signal-contingent choices from A.

In the representation, each $\succsim^A$ ranks experiments by their expected utility under prior ν (full support), utility index v, and correct forecasting of Receiver's choices. In particular,

$$V^A(\sigma) := \max \sum_{\omega \in \Omega} \nu_\omega \sum_{s \in \sigma} s_\omega v(f^s_\omega) \text{ subject to } f^s \in c^s(A) \tag{1}$$

is Sender's *value of information σ at menu A*, where ν_ω is Sender's prior probability of state ω and $v : X \to \mathbb{R}$ his utility index.[3] This is analogous to an indirect utility function for the sender in Bayesian Persuasion models (Kamenica and Gentzkow, 2011), where ties—non-singleton sets $c^s(A)$—are resolved in a standard "Sender-preferred" manner.

The axioms characterizing representation (1) employ both informational preferences $\succsim^A$ and signal-contingent choices c^s. Familiar Independence and Continuity axioms are defined using an appropriate mixture operation on the space of experiments, and the Anscombe-Aumann State Independence axiom is expressed using both preferences $\succsim^A$ and choices c^s. The key axiom, *Consistency*, states that Sender only cares about the state-contingent distributions over outcomes generated by the information structure and Receiver's choices. This ensures Sender's prior ν and utility index v are not menu dependent.

The representation theorems establish uniqueness of all parameters (ν, v, μ, and u) given Sender's preferences for information and Receiver's signal-contingent choices. It turns out, however, that all parameters can be identified from Sender's preferences for information (Theorem 3). Section 4 describes the steps required to elicit these parameters.

Sender's preferences can also be used to compare the attributes of the agents. I show in Section 5 that v and u are more aligned if perfect information is Sender's most-preferred experiment in a larger class of menus called *bets*.[4] In the limit, when these indices coincide, perfect information is Sender's most-preferred information structure in all bets. Similarly, priors are more aligned if more signals result in agreement regarding the posterior ranking of arbitrary events. "Extreme" experiments are composed of such signals and make Sender's preferences locally monotone with respect to the Blackwell ordering in bets. In the limit, priors coincide and Sender's preferences are globally monotone with respect to the Blackwell ordering in bets. Combining these limit cases establishes that Sender and Receiver share

[3] The statement '$s \in \sigma$' means s is a column of σ. For lotteries p, let $v(p) := \sum_x v(x)p(x)$.

[4] A bet is a menu $A = \{f, g\}$ where there exist lotteries $p, q \in \Delta X$ such that $f_\omega, g_\omega \in \{p, q\}$ for all ω.

a common prior and utility index if and only if Sender's preferences satisfy the Blackwell ordering in all bets, which implies Sender's preferences satisfy the Blackwell ordering in all menus. Thus, in the behavioral interpretation of the model, dynamically consistent behavior is characterized by adherence to the Blackwell information ordering.

These results illustrate the power of information structures as objects of choice. Preferences for information may seem rather abstract, but it is not difficult to see how individuals might reveal them. For example, many online retailers enable custom tailoring of information about new products or services. By customizing such news feeds, individuals reveal what type of information they consider to be the most valuable—and the available actions (product choices) are also observed, as in my framework. Preferences for information, conditioned on choice sets, can also be elicited in laboratory settings. This paper does not carry out any empirical or experimental exercises, but shows that informational choice may be a valuable tool for analysts interested in testing models or identifying parameters.

Finally, preferences for information are a natural primitive in each interpretation of the model. In Bayesian Persuasion settings, Sender's informational preferences, together with Receiver's signal-contingent choices, are the most an analyst can hope to observe. In the behavioral interpretation, informational choice offers an effective form of commitment power: those unable to constrain their choice sets may resist temptation when it arrives by selectively paying attention to information—in particular, to sources that are more likely to make tempting alternatives seem less appealing.

1.1 Related Literature

This paper is related to the growing literature on information disclosure with sender commitment power initiated by Kamenica and Gentzkow (2011), henceforth KG, and Rayo and Segal (2010).[5] My model is most closely related to the framework of KG, where a sender chooses an experiment and a receiver takes an action after observing a signal generated by the experiment. Building on techniques of Aumann and Maschler (1995), KG study when and how the sender can improve his own expected payoff through "persuasion": choosing an experiment and committing to revealing its signal.

My analysis and motivation differs from that of KG in several ways. Rather than studying when the sender might benefit from persuasion, I examine how observed choices can be used to test whether the sender and receiver conform to the KG framework. My representation characterizes what it means for the receiver to be a Bayesian information processor and the sender a sophisticated planner. The characterization is expressed in terms of the choices

[5] In contrast, cheap talk models (Crawford and Sobel, 1982) assume no commitment power.

agents actually make in the KG framework—the sender chooses information, and the receiver chooses among risky alternatives. I also show how the sender's choices can be used to identify and compare the beliefs and utilities of the agents. While KG take as given a fixed set of actions and consider a sender who is free to choose his most-preferred information structure, my analysis involves a rich set of choice data: for each menu of acts, the sender's full ranking of information structures is observed. The full ranking is needed to characterize the agents and identify parameters. Finally, while KG assume common priors, my framework permits the sender and receiver to hold different priors.[6]

The behavioral interpretation of my model offers a different perspective on temptation and commitment. Since Kreps (1979), preferences for flexibility or commitment are typically modeled as preferences over menus of alternatives. Such preferences represent hard commitment in that menus indicate the options available for later consumption. Utilizing preferences over menus of lotteries, Dekel, Lipman, and Rustichini (2001) characterize a representation driven by a set of subjective states, extending the representation of Kreps (1979). In a similar setting, Gul and Pesendorfer (2001) characterize a representation where an agent faces temptation and suffers a cost of self-control. Gul and Pesendorfer's model nests a generalization of Strotz (1955). My representation can be interpreted as a Strotz model where the agent controls information, but not the actions, available to his future self.

Behavioral economists have developed models where agents regulate behavior through information suppression or self-signaling. Carrillo and Mariotti (2000) show that, in a model of personal equilibrium, time-inconsistent agents may benefit from acquiring less information, while Grant, Kajii, and Polak (2000) examine when a dynamically consistent individual with non-expected utility preferences prefers more information to less. Benabou and Tirole (2002, 2006) study equilibrium models where players rationally limit information available to future selves. In the persuasion literature, Lipnowski and Mathevet (2018) examine how a benevolent principal should disclose information to agents who are susceptible to temptation, reference dependence, or other behavioral phenomena. Similarly, the behavioral interpretation of my model provides a general analysis of the incentives for information acquisition for individuals lacking time-consistent preferences or prior beliefs.

Azrieli and Lehrer (2008) consider preferences over information structures and provide necessary and sufficient conditions for such a preference to be represented by expected utility in some decision problem.[7] In their representation, with the prior taken as given, a utility index and menu of actions are deduced from the preference for information but cannot be

[6] Alonso and Câmara (2016) extend the KG framework to allow heterogeneous priors and find that (generically) the sender benefits from persuasion under heterogeneous priors.

[7] See also Gilboa and Lehrer (1991), who study a similar problem for partitional information structures.

uniquely pinned down. Azrieli and Lehrer (2008) note that their axioms can be modified to allow an endogenous prior but that it, too, cannot be uniquely identified. My model circumvents these identification issues by examining preferences for information in all exogenously specified menus—even with time-inconsistent priors or utilities, all parameters can be uniquely identified from this richer collection of preferences.

Several authors have studied Bayesian updating from a decision-theoretic perspective. Ghirardato (2002) develops a representation using conditional preferences over acts; that is, families of preferences indexed by events, with the interpretation that the event represents an observed signal. Karni (2007) uses a similar family of conditional preferences defined over conditional acts. The extra structure of conditional acts permits both prior beliefs and state-dependent utilities to be identified, in addition to testing Bayesian updating of partitional information. Wang (2003) axiomatizes Bayes' rule and some of its extensions in a setting with conditional preferences over infinite-horizon consumption-information profiles; preferences are conditioned on sequences of previously realized events. My representation characterizes Bayesian updating using signal-contingent preferences over standard Anscombe-Aumann acts. The set of signals is richer than the state space over which acts are defined, enabling a simple and intuitive characterization.

Finally, Lu (2016) shows how random choice data reveals an individual's information, provided the individual is a Bayesian expected utility maximizer. Decision-theoretic models of rational inattention[8] also use standard choice primitives to make inferences about an individual's preferences, beliefs, and information processing ability. My framework uses informational choice to make inferences about one's underlying tastes and beliefs.

2 Framework and Notation

2.1 Outcomes, lotteries, acts

Let X denote a finite set of $N \geq 2$ *outcomes*, with generic members denoted x, y. Elements of ΔX, *lotteries*, are denoted p, q.[9] A lottery p assigns probability $p(x)$ to outcome x.

A *utility index* is a function $u : X \to \mathbb{R}$. If $p \in \Delta X$ and u is a utility index, let $u(p) := \sum_{x \in X} u(x)p(x)$ denote the expected utility of p. The notation $u' \approx u$ indicates that u' is a positive affine transformation of u.

Let Ω denote a finite, exogenous set of $W \geq 2$ states. Arbitrary states are typically denoted ω, ω', while members of $\Delta\Omega$, *probability distributions* over Ω, are denoted μ or ν.

[8] See Denti, Mihm, de Oliveira, and Ozbek (2016), Ellis (2018), and Caplin and Dean (2015).

[9] For finite sets Y, ΔY denotes the probability simplex over Y, equipped with the usual mixture operation.

As a notational convention, subscripts denote states. For example, a distribution $\mu \in \Delta\Omega$ may be expressed as $\mu = (\mu_\omega)_{\omega\in\Omega}$, where μ_ω is the probability assigned to state ω.

A function $f : \Omega \to \Delta X$ is an Anscombe-Aumann *act*. Let F denote the set of all acts. Acts are typically denoted f, g, h, and may be written as profiles: $f = (f_\omega)_{\omega\in\Omega}$, where $f_\omega \in \Delta X$. The set F is equipped with the standard mixture operation: if $f, g \in F$ and $\alpha \in [0, 1]$, then $\alpha f + (1 - \alpha)g := (\alpha f_\omega + (1 - \alpha)g_\omega)_{\omega\in\Omega}$.

A *menu* is a finite, nonempty set of acts. Menus are typically denoted A, B. Let $\mathcal{A}$ denote the set of all menus. Both F and ΔX are endowed with the standard Euclidean metric, and $\mathcal{A}$ with the associated Hausdorff metric.

2.2 Blackwell Experiments

Definition 1 (Blackwell Experiment)**.** A matrix σ with entries in $[0, 1]$ is a (finite) *Blackwell experiment* if it has exactly W rows, no columns consisting only of zeros and, for each row, the sum of entries is exactly one. Let $\mathcal{E}$ denote the set of all Blackwell experiments.

In an experiment σ, columns represent signals that may be generated, and rows state-contingent probability distributions over such signals. The requirement that each column contains at least one non-zero entry eliminates signals that have zero probability of occurrence in every state. Note that entries in any given column are not required to sum to one.

It will be convenient to express experiments in terms of their columns. Let $S := \{s = (s_\omega)_{\omega\in\Omega} \in [0, 1]^\Omega : \exists\omega \text{ such that } s_\omega \neq 0\}$. Elements of S are called *signals*. Every column of an experiment σ corresponds to a signal s where s_ω is the entry for the column in row ω.[10]

The statement '$s \in \sigma$' means s is a column of σ. Experiments may have duplicate columns; thus, when quantifying over signals in an experiment, different columns are distinguished even if they are duplicates. For example, the requirement that each row in σ has entries summing to one may be expressed as '$\forall\omega$, $\sum_{s\in\sigma} s_\omega = 1$' because the summation notation implicitly distinguishes between duplicate columns of σ. Similarly, statements like '$\forall s \in \sigma$, $y^s \in Y$' associate potentially different members of Y to different columns of σ, even if those columns are duplicates.

If $\sigma \in \mathcal{E}$ and $\alpha \in (0, 1)$, let $\alpha\sigma$ denote the matrix formed by multiplying each entry of σ by α. If $\sigma, \sigma' \in \mathcal{E}$ and $\alpha \in (0, 1)$, then $\alpha\sigma \cup (1 - \alpha)\sigma'$ denotes the matrix formed by appending $(1 - \alpha)\sigma'$ to the right of $\alpha\sigma$. This mixture yields a well-defined experiment, but the operation is not commutative. Intuitively, $\alpha\sigma \cup (1 - \alpha)\sigma'$ is the information structure

[10] The term "signal" refers to both a message that might be generated and the state-contingent likelihoods of that message. These likelihoods, together with a prior, are sufficient to compute a Bayesian posterior.

coming about if nature randomly selects between σ and σ' (with probabilities α and $1-\alpha$) before using the chosen matrix to generate a signal.[11]

Additional mixture operations on acts can be defined using the notation of signals and experiments. If $f, g \in F$ and $s \in S$, let $sf + (1-s)g := (s_\omega f_\omega + (1-s_\omega)g_\omega)_{\omega\in\Omega}$. More generally, if $\sigma \in \mathcal{E}$ and $f^s \in F$ for all $s \in \sigma$, let $\sum_{s\in\sigma} sf^s := \left(\sum_{s\in\sigma} s_\omega f^s_\omega\right)_{\omega\in\Omega}$. This is the act formed by applying weight s_ω to f^s_ω $(s \in \sigma)$ in state ω.

An ε-neighborhood of s is given by $N^\varepsilon(s) := \left\{t : d\left(\frac{s}{\|s\|}, \frac{t}{\|t\|}\right) < \varepsilon\right\}$, where d is the Euclidean metric and $\|\cdot\|$ the Euclidean norm. An ε-neighborhood of σ is given by $N^\varepsilon(\sigma) := \left\{\sigma' : t' \in \bigcup_{s\in\sigma} N^\varepsilon(s)\ \forall t' \in \sigma'\right\}$. Thus, signals are close if their associated Bayesian posteriors are close (in a Euclidean sense) for all priors, and experiments are close if their constituent signals are close. The same notation $N^\varepsilon(\cdot)$ is used for neighborhoods around signals and experiments, but the type of neighborhood will be clear from context.

2.3 Primitives

I take as primitive two collections of choice data:

1. For each menu $A \in \mathcal{A}$, a preference $\succsim^A$ over $\mathcal{E}$. Let $\succsim = (\succsim^A)_{A\in\mathcal{A}}$.
2. For each signal $s \in S$, a choice correspondence c^s such that, for each menu A, $c^s(A)$ is a nonempty subset of A. Let $c = (c^s)_{s\in S}$.

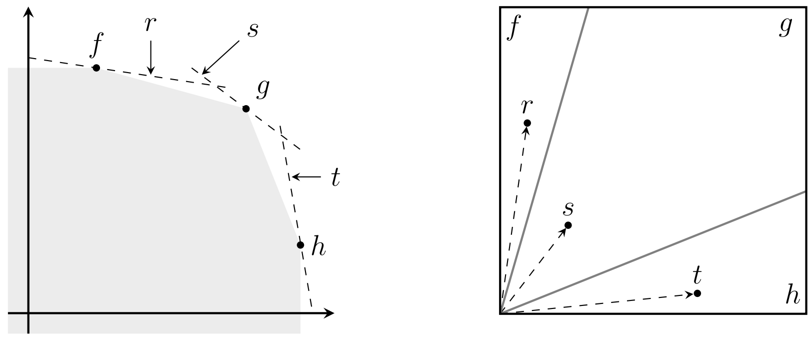
The family $\succsim = (\succsim^A)_{A\in\mathcal{A}}$ captures Sender's preferences for information. In particular, $\sigma \succsim^A \sigma'$ means Sender prefers to expose Receiver to information σ rather than σ' given that Receiver chooses from A after observing a signal. The collection $c = (c^s)_{s\in S}$ captures Receiver's signal-contingent choices. Specifically, $c^s(A)$ contains the acts chosen by Receiver from A after observing signal s. In practice, Receiver's choice is conditioned on a pair (σ, s) where $s \in \sigma$ because a signal must be generated by some experiment. However, for a Bayesian information processor, only the entries of s (not the other columns of σ) matter. To minimize notation, I condition choices on signals s instead of pairs (σ, s).

3 The Representation

3.1 Value of Information and Bayesian Representations

The goal is to represent Sender as a sophisticated Bayesian planner and Receiver as a Bayesian information processor. In the representation, each agent is an expected utility maximizer, but Sender and Receiver need not have a common prior or utility function.

[11] See Birnbaum (1961) or Torgersen (1977) for more general analysis of mixtures of experiments.



(a) Utility Space (b) Signal space

Figure 1: Geometric representations of Receiver's behavior when $|\Omega| = 2$. Receiver prefers f over g at signal $\tilde{s}$ if and only if $\sum_\omega u(f_\omega)\tilde{s}_\omega\mu_\omega \geq \sum_\omega u(g_\omega)\tilde{s}_\omega\mu_\omega$. Thus, in utility space, acts f correspond to points $(\mu_1 u(f_1), \mu_2 u(f_2))$, and choices at signals $\tilde{s}$ are determined by the ratio $\tilde{s}_1/\tilde{s}_2$. Consequently, Receiver's choices from $A = \{f, g, h\}$ partition S into convex cones. Arrows pointing to signals in (b) are perpendicular to the corresponding lines in (a).

Definition 2 (Value of Information Representation)**.** A pair (ν, v) is a *Value of Information Representation* for $(\succsim, c)$ if $\nu \in \Delta\Omega$ has full support, $v : X \to \mathbb{R}$ is non-constant and, for each menu A, $\succsim^A$ is represented by the function

$$V^A(\sigma) := \max \sum_{\omega\in\Omega} \nu_\omega \sum_{s\in\sigma} s_\omega v(f^s_\omega) \text{ subject to } f^s \in c^s(A). \tag{2}$$

In a Value of Information Representation, Sender correctly forecasts the signal-contingent choices made by Receiver and assigns expected utility to experiments σ using utility function v and subjective prior ν; ties (non-singleton choices $c^s(A)$) are resolved in a standard Sender-preferred manner. Definition 2 does not make any assumptions about Receiver's behavior, but the standard model involves a Bayesian Receiver:

Definition 3 (Bayesian Representation)**.** A pair (μ, u) is a *Bayesian Representation* for c if $\mu \in \Delta\Omega$ has full support, $u : X \to \mathbb{R}$ is non-constant and, for all $s \in S$ and $A \in \mathcal{A}$,

$$c^s(A) = \left\{ f \in A \;:\; \forall g \in A,\; \sum_\omega u(f_\omega)\mu^s_\omega \geq \sum_\omega u(g_\omega)\mu^s_\omega \right\} \tag{3}$$

where the posteriors μ^s satisfy Bayes' rule: $\forall \omega \in \Omega,\ \mu^s_\omega = \frac{\mu_\omega s_\omega}{\sum_{\omega'\in\Omega}\mu_{\omega'}s_{\omega'}}$.

In a Bayesian Representation, choices $c^s(A)$ maximize expected utility given prior μ, utility index u, and Bayesian updating: upon observing signal s, Receiver updates his prior μ to the Bayesian posterior μ^s, then chooses $f \in A$ if and only if f maximizes expected utility under beliefs μ^s. Figure 1 provides geometric representations of this behavior.

3.2 Characterization of Sender

This section characterizes Sender's behavior. The first three axioms are essentially the von Neumann-Morgenstern axioms, adapted to operate on experiments and their mixtures.

Axiom 1.1 (Rationality). Each $\succsim^A$ is complete and transitive.

Axiom 1.2 (Independence). If $\sigma \succ^A \sigma'$ and $\alpha \in (0,1)$, then $\alpha\sigma \cup (1-\alpha)\sigma'' \succ^A \alpha\sigma' \cup (1-\alpha)\sigma''$ for all σ''.

Axiom 1.3 (Continuity). If $\sigma \succ^A \sigma' \succ^A \sigma''$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha\sigma \cup (1-\alpha)\sigma'' \succ^A \sigma' \succ^A \beta\sigma \cup (1-\beta)\sigma''$.

When ranking experiments, Sender anticipates that Receiver makes choices using the correspondences c^s. Therefore, the remaining axioms describe properties of Sender's preferences in anticipation of Receiver's choices. Axioms 1.5 and 1.6 condition preferences $\succsim^A$ on c, while Axioms 1.4 and 1.7 involve joint restrictions on $\succsim$ and c.

For each menu A, let $\mathcal{E}^c(A)$ denote the set of experiments σ such that, for all $s \in \sigma$, $c^s(A)$ is a singleton. The next axiom is needed to disentangle Sender's beliefs and utilities.

Axiom 1.4 (Non-Degeneracy). There exist $A \in \mathcal{A}$ and $\sigma, \sigma' \in \mathcal{E}^c(A)$ such that $\sigma \succ^A \sigma'$.

If $\sigma \in \mathcal{E}^c(A)$, let $c^s(A) \in F$ denote the unique act chosen by Receiver at s, and $c^s_\omega(A) \in \Delta X$ the lottery specified by $c^s(A)$ in state ω. Then $c^\sigma(A) := \sum_{s\in\sigma} s c^s(A) \in F$ is the *induced act* for σ at A. When using the notation $c^\sigma(A)$, it is implicit that $\sigma \in \mathcal{E}^c(A)$. Intuitively, $c^\sigma(A)$ is the "average" act chosen by Receiver when signals are generated by σ. It is formed by applying weight s_ω to $c^s_\omega(A)$ in state ω, so that the state-contingent distributions over signals given by σ yield state-contingent mixtures of lotteries. Each state-contingent mixture of lotteries is reduced to a single lottery, yielding an Anscombe-Aumann act.

Axiom 1.5 (Consistency). If $c^\sigma(A) = c^{\hat\sigma}(B)$, $c^{\sigma'}(A) = c^{\hat\sigma'}(B)$, and $\sigma \succsim^A \sigma'$, then $\hat\sigma \succsim^B \hat\sigma'$.

Consistency states that induced acts, when they exist, determine Sender's ranking of experiments. Thus, Sender correctly forecasts Receiver's choices and only cares about the distribution of outcomes in each state of the world—not the particular combination of menu and experiment giving rise to those distributions.

For $p \in \Delta X$, $h \in F$, and $\omega \in \Omega$, let $p[\omega]h$ denote the act formed by taking h and replacing h_ω with p. The next axiom is analogous to the State Independence axiom in the

Anscombe-Aumann model, once again adapted to operate on experiments.[12]

Axiom 1.6 (State Independence). Suppose $c^{\sigma}(A) = p[\omega]h$ and $c^{\sigma'}(A) = q[\omega]h$ while $c^{\hat{\sigma}}(A) = p[\omega']\hat{h}$ and $c^{\hat{\sigma}'}(A) = q[\omega']\hat{h}$. Then $\sigma \succsim^A \sigma'$ implies $\hat{\sigma} \succsim^A \hat{\sigma}'$.

When Receiver is Bayesian, Axioms 1.1–1.6 are sufficient to establish a unique Value of Information Representation (ν, v) for the restriction of $\succsim^A$ to $\mathcal{E}^c(A)$ (see Theorem 1A in the appendix). Since ν and v are uniquely identified from this subset of the primitives, there is a good deal of freedom in extending the representation to menu-experiment pairs where Receiver is tied at some signal realization. To impose the standard "Sender-preferred" tie-breaking rule, one final axiom—and some additional notation—is required.

A sequence of menus $(A^n)_{n=1}^{\infty}$ *converges in choice to* A, denoted $A^n \to^c A$, if $A^n \to A$ and, for all s, $c^s(A^n)$ converges to a singleton in the Hausdorff metric. The idea is that while $c^s(A)$ may contain multiple acts, small perturbations A^n of A eliminate the tie, making $c^s(A^n)$ a singleton. Other signals might yield ties at A^n but, in the limit, ties are eliminated for all s. This way, a sequence $A^n \to^c A$ corresponds to a tie-breaking selection where the choice at s is $c^s(A^\infty) := \lim c^s(A^n)$. When Receiver is Bayesian, $c^s(A^\infty)$ is an element of $c^s(A)$. Thus, if $A^n \to^c A$, every experiment σ has an associated *limit induced act* $c^\sigma(A^\infty) := \sum_{s\in\sigma} s c^s(A^\infty)$, which is the induced act coming about if Receiver chooses $c^s(A^\infty)$ from A at s.

Sender-preferred tie-breaking amounts to $\succsim^A$ being determined by a tie-breaking selection (hence, a sequence $A^n \to^c A$) where, for every σ, the associated induced act cannot be improved upon by changing the selection. The difficulty is that Sender's ranking of acts is not directly observed, but must be inferred from informational preferences $\succsim^A$.

Let $\succcurlyeq^*$ be a relation on F such that $f \succcurlyeq^* g$ if there exists $A \in \mathcal{A}$, $\sigma, \sigma' \in \mathcal{E}^c(A)$, $h \in F$, and $\alpha \in (0,1)$ such that $c^\sigma(A) = \alpha f + (1-\alpha)h$, $c^{\sigma'}(A) = \alpha g + (1-\alpha)h$, and $\sigma \succsim^A \sigma'$. The relation $\succcurlyeq^*$ captures Sender's ranking of f and g from the ranking of experiments giving rise to induced acts of the form $\alpha f + (1-\alpha)h$ and $\alpha g + (1-\alpha)h$ in some menu.

Axiom 1.7* (Sender Optimism*). For every A, $\sigma \succsim^A \sigma'$ if and only if for all $B^n \to^c A$, there exists $A^n \to^c A$ such that $c^\sigma(A^\infty) \succcurlyeq^* c^{\sigma'}(B^\infty)$.

This axiom states that $\succsim^A$ is determined by a Sender-optimal selection. Specifically, $\sigma \succsim^A \sigma'$ if for every limit induced act $g = c^{\sigma'}(B^\infty)$ there exists a limit induced act $c^\sigma(A^\infty)$ that Sender prefers to g, making the best-case selection for σ more attractive than the best-case selection for σ'. The axiom may equivalently be expressed as:

[12]The standard axiom says: if ω, ω' are non-null and $p[\omega]h$ is weakly preferred to $q[\omega]h$, then $p[\omega']\hat{h}$ is weakly preferred to $q[\omega']\hat{h}$ for all $\hat{h}$. Axiom 1.6 rules out null states, so that ν has full support.

Axiom 1.7 (Sender Optimism). For every A, $\sigma \succsim^A \sigma'$ if and only if for all $B^n \to^c A$, there exists $A^n \to^c A$, $\hat{A} \in \mathcal{A}$, $\hat{\sigma}, \hat{\sigma}' \in \mathcal{E}^c(\hat{A})$, $h \in F$, and $\alpha \in (0,1)$ such that $c^{\hat{\sigma}}(\hat{A}) = \alpha c^{\sigma}(A^\infty) + (1-\alpha)h$, $c^{\hat{\sigma}'}(\hat{A}) = \alpha c^{\sigma'}(B^\infty) + (1-\alpha)h$, and $\hat{\sigma} \succsim^{\hat{A}} \hat{\sigma}'$.

The only difference between Axioms 1.7 and 1.7* is that Axiom 1.7 does not invoke the relation $\succcurlyeq^*$—only the primitives $(\succsim, c)$ are used. Either way, the idea is that $\succsim^A$ is determined by a Sender-optimal tie-breaking selection. When Receiver is Bayesian, $\succcurlyeq^*$ is complete and there are many sequences that converge in choice to A. Thus, while Axiom 1.7/1.7* technically makes joint restrictions on $\succsim$ and c, it does not impose any structure on c that is not already satisfied by a Bayesian.

Theorem 1. *Suppose c has a Bayesian Representation. Then $(\succsim, c)$ satisfies Axioms 1.1–1.7 if and only if it has a Value of Information Representation. Moreover, ν is unique and v is unique up to positive affine transformation.*

Theorem 1 states that Axioms 1.1–1.7 are necessary and sufficient for the existence of a Value of Information Representation provided c has a Bayesian Representation. This is an intermediate step toward a complete representation theorem; the next section provides a characterization of Bayesian Representations, completing the overall representation.

Although most of the axioms are straightforward adaptations of the Anscombe-Aumann axioms, Theorem 1 is not a direct corollary of the Anscombe-Aumann theorem. There are two obstacles. First, it is not obvious that varying σ generates enough variation in Receiver's choices to establish existence of an expected utility representation for any $\succsim^A$. A key step of the proof constructs a menu A^* from which existence is established and candidates for ν and v can be identified. Second, it is also not obvious that the Consistency axiom is strong enough to ensure menu-independent beliefs and utilities can be obtained for Sender. If two menus give rise to disjoint sets of induced acts, then Consistency seemingly has no bite and Sender could hold different beliefs and/or utilities in those menus. The main challenge of the proof is to show that any two menus can be connected by a finite sequence of menus with significantly overlapping sets of induced acts along the way, thus ensuring uniqueness.

As explained above, the only role of Axiom 1.7 is to handle ties. Since (ν, v) can be identified from data points not involving ties, alternative tie-breaking rules can be imposed by replacing Axiom 1.7 with different axioms. The Sender-preferred solution, however, is standard in the literature and simplifies subsequent analysis of the model (sections 4 and 5).

3.3 Characterization of Receiver

The final axioms characterize Receiver as a standard Bayesian; unlike the first set of axioms, these involve only Receiver choice data c. If $\alpha \in [0,1]$ and $A, B \in \mathcal{A}$, let $\alpha A + (1-\alpha)B := \{\alpha f + (1-\alpha)g : f \in A, g \in B\}$. If $L \subseteq \Delta X$ is finite, $h \in F$, and $\omega \in \Omega$, let $L[\omega]h := \{p[\omega]h : p \in L\}$.

Axiom 2.1 (Standard Receiver Preferences).

(i) Each c^s satisfies WARP: if $f, g \in A \cap B$, $f \in c^s(A)$, and $g \in c^s(B)$, then $f \in c^s(B)$.

(ii) For every s, there exists A such that $c^s(A) \neq A$.

(iii) If $A, B \in \mathcal{A}$, $\alpha \in [0,1]$, and $s \in S$, then $c^s(\alpha A + (1-\alpha)B) \subseteq \alpha c^s(A) + (1-\alpha)c^s(B)$.

(iv) Each c^s is upper hemicontinuous.

(v) If $L \subseteq \Delta X$ is finite, $h, h' \in F$, and $s_\omega, s'_{\omega'} > 0$, then $c^s_\omega(L[\omega]h) \subseteq c^{s'}_{\omega'}(L[\omega']h')$.

Axiom 2.1 states five properties required for each c^s to be rationalized by subjective expected utility. Parts (i) and (ii) imply each c^s has a non-degenerate rationalizing preference $\succsim^s$, while (iii) and (iv) ensure $\succsim^s$ satisfies standard Independence and Continuity properties. Finally, part (v) implies $\succsim^s$ satisfies a version of the State Independence axiom; in particular, Receiver ranks lotteries independently of the state and signal realization.

If $A \in \mathcal{A}$, $s \in S$, and $h \in F$, let $sA + (1-s)h := \{sf + (1-s)h : f \in A\}$.

Axiom 2.2 (Bayesian Consistency). If $tf + (1-t)h \in c^s(tA + (1-t)h)$, then $sf + (1-s)h' \in c^t(sA + (1-s)h')$.

Axiom 2.2 states that signal likelihoods are exchangeable with outcome probabilities. When comparing acts in $tA + (1-t)h$, an expected utility maximizer effectively ignores the $(1-t)h$ component, resulting in a comparison between acts of A whose outcomes have been scaled by signal t. Bayesian Consistency states that making such comparisons after observing s is equivalent to comparing acts in $sA + (1-s)h$ after observing t.

Theorem 2. *Choices c satisfy Axioms 2.1–2.2 if and only if they have a Bayesian Representation (μ, u). Furthermore, μ is unique and u is unique up to positive affine transformation.*

The proof of Theorem 2 is quite simple. By Axiom 2.1, there exists a utility index u such that c^s is rationalized by expected utility with beliefs μ^s and utility u. Axiom 2.2 ensures μ^s is the Bayesian posterior of $\mu := \mu^e$, where $e = (1, \ldots, 1) \in S$ is an uninformative signal.

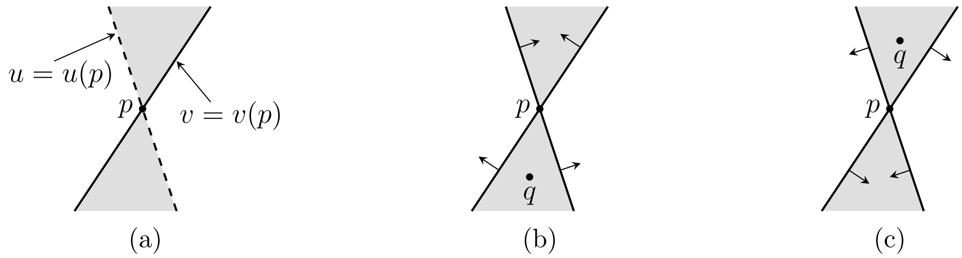


Figure 2: The (linear) indifference curve for v through p is revealed by $\succsim$, as is the agreement region (shaded) for u and v; thus, indifference curves for u can be identified (panel (a)). Panels (b) and (c) indicate the two possibilities for the direction of increasing utilities that are consistent with the agreement region. In each case, both agents strictly prefer p over q.

4 Identification

Theorems 1 and 2 characterize Sender and Receiver in terms of the primitives $(\succsim, c)$ and establish that ν, μ, v, and u can be identified from those primitives. This section provides a stronger identification result: all four parameters can be identified using only $\succsim$.

Definition 4. Parameters (ν, μ, v, u) *represent* $\succsim$ if there exists $c = (c^s)_{s \in S}$ such that $(\succsim, c)$ has a Value of Information Representation (ν, v) and c has a Bayesian Representation (μ, u).

Intuitively, parameters (ν, μ, v, u) represent $\succsim$ if Sender has prior ν, utility index v, and believes Receiver's choices c are governed by a Bayesian Representation (μ, u).

Theorem 3. *If (ν, μ, v, u) and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$ represent $\succsim$, then $\nu = \dot{\nu}$, $\mu = \dot{\mu}$, $v \approx \dot{v}$, and $u \approx \dot{u}$.*

Theorem 3 states that Sender's preferences $\succsim$, alone, are sufficient to pin down the priors ν, μ and utility indices v, u; for identification purposes, Receiver's choices c are not required.[13] Since Definition 4 only references $\succsim$, the elicited parameters (μ, u) capture Sender's beliefs about Receiver; to determine whether those beliefs are correct, data c would be required.

Some additional notation and terminology is required to sketch the proof. A *bet* is a menu of the form $A = \{pEq, pFq\}$ where $E \not\subseteq F$ and $F \not\subseteq E$. When the need to be explicit about E and F arises, I refer to such menus as *EF-bets*. Similarly, a bet may be referred to as a *pq-bet*. Receiver's choices from bets $A = \{pEq, pFq\}$ only depend on his ranking of p and q and his posterior ranking of E and F. For example, if $u(p) > u(q)$, then $pEq \in c^s(A)$ if and only if $\mu^s(E) \geq \mu^s(F)$. Hence, cardinal properties of u do not influence choice in bets.

For each ω, let e^ω denote the signal s such that $s_\omega = 1$ and $s_{\omega'} = 0$ for all $\omega' \neq \omega$. The identity matrix, $\sigma^* := [e^\omega : \omega \in \Omega]$, denotes *perfect information*: σ^* reveals the true state ω. The signal $e = (1, \dots, 1) \in S$ qualifies as an experiment representing *no information*.

[13] This result suggests it may be possible to characterize the model using only Sender's preferences $\succsim$ as primitive. I do not pursue this here, but consider it an interesting avenue for future research.

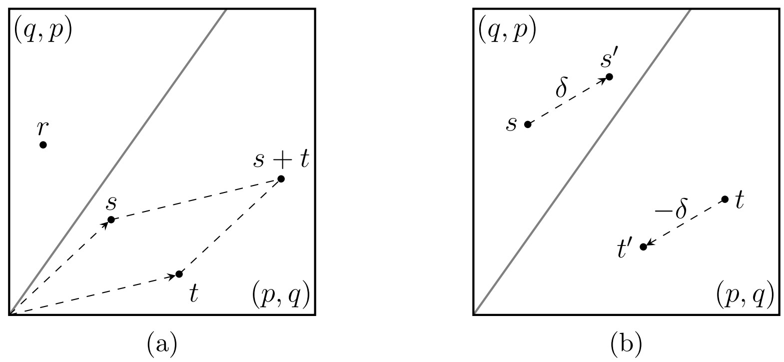


Figure 3: In (a), s and t are equivalent because they each make Receiver choose (p, q) from $A = \{(p, q), (q, p)\}$. Thus, so does $s + t$, so that $[r, s, t] \sim^A [r, s + t]$. The line separating the (p, q) and (q, p) regions has slope $\frac{\mu_1}{\mu_2}$. Thus, knowing which signals are equivalent reveals μ. In (b), $[s, t] \sim^A [s + \delta, t - \delta]$ implies $\frac{\delta_1}{\delta_2} = \frac{\nu_2}{\nu_1}$, revealing ν.

The proof of Theorem 3 involves three steps. First, observe that for any pq-bet A, σ^* is top-ranked by $\succsim^A$ if and only if v and u agree on the ranking of p and q in that $[v(p) \geq v(q)$ and $u(p) \geq u(q)]$ or $[v(q) \geq v(p)$ and $u(q) \geq u(p)]$; Sender prefers perfect information in such cases because it makes Receiver choose Sender's most-preferred act in every state. Thus, by fixing p and varying q, $\succsim$ reveals an "agreement region" bounded by linear indifference curves for v and u through p (Figure 2). The indifference curve for v is revealed by the set of pq-bets A where $\succsim^A$ is degenerate, forcing the indifference curve for u to be the other plane bounding the agreement region. These curves narrow the possibilities for v and u down to two cases: there are indices v and u such that either (v, u) or $(-v, -u)$ are the correct functions.

The second step elicits μ and ν. Let $A = \{pEq, pFq\}$ be a bet where neither agent is indifferent between p and q. To identify μ, it will suffice to identify which signals make Receiver rank E strictly more likely than F. Such *equivalent signals* lead Receiver to make the same choice in any EF-bet, so that merging them does not change Sender's value of information. For example, if $s, t \in \sigma$ each make Receiver choose pEq, then so does $s+t$. Thus, equivalent signals are revealed by Sender's ranking of experiments of the form $\sigma = [r, s, t]$ and $\sigma = [r, s + t]$, allowing μ to be identified (see Figure 3a). To identify ν, once again let $A = \{pEq, pFq\}$ be a bet where neither agent is indifferent between p and q. Knowing μ, one can construct experiments $\sigma = [s, t]$ and $\sigma^\delta = [s + \delta, t - \delta]$ where (i) s and t result in different Receiver choices, and (ii) $\delta \in \mathbb{R}^\Omega$ is sufficiently small that Receiver's choice at $s + \delta$ coincides with that at s and his choice at $t - \delta$ coincides with that at t. By eliciting δ such that $\sigma \sim^A \sigma^\delta$, one can deduce ν (see Figure 3b).

The final step determines which pair of utility indices—(v, u) or $(-v, -u)$—is correct.

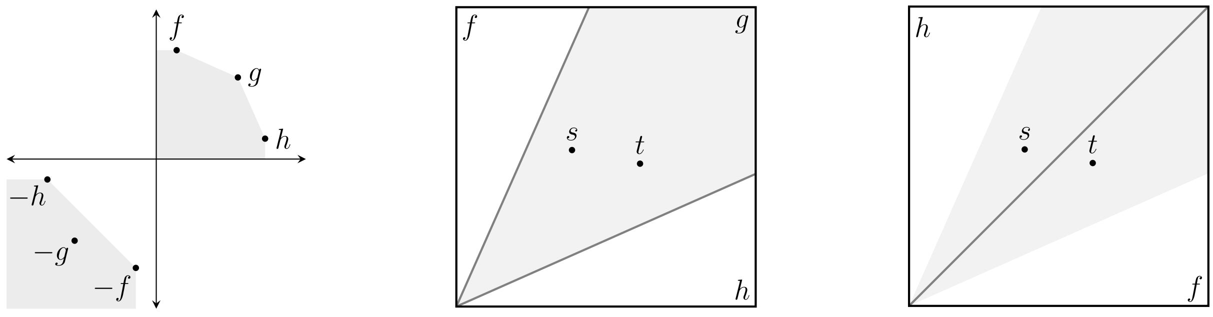


Figure 4: Identifying v and u. If f, g, and h are chosen by Receiver under (v, u), then only f and h are chosen under $(-v, -u)$ (panel (a)). Thus, u and $-u$ yield different divisions of S (panels (b) and (c)). In (b), Sender ranks $\sigma = [s, t]$ indifferent to e because both signals result in choice g. In (c), s and t yield different choices, so that σ is not ranked indifferent to e. Thus, only one of (v, u) or $(-v, -u)$ can be consistent with Sender's preferences.

Consider a menu $A = \{f, g, h\}$ where, under (μ, u), each act is the unique optimum at some signal. Then, under $(\mu, -u)$, one of the acts is not chosen at any signal. This yields a different division of S, so that there are experiments σ, σ' that Sender ranks indifferent under $(-v, -u)$ but not under (v, u); see Figure 4.

5 Comparing Parameters

In this section, I show how Sender's preferences $\succsim$ can be used to compare the priors and utilities of the agents. First, I provide a definition of "more-aligned" utility indices. I characterize this relation, as well as the limit case $v \approx u$, in terms of Sender's preference for information. I then conduct a similar exercise for the priors.

Some results involve the Blackwell information ordering, denoted $\sqsupseteq$. This is a partial order on $\mathcal{E}$ where $\sigma \sqsupseteq \sigma'$ if and only if σ' is a *garbling* of σ; that is, $\sigma' = \sigma M$, where each row of the matrix M is a probability distribution.[14] Clearly, $\sigma^* \sqsupseteq \sigma \sqsupseteq e$ for all σ, where σ^* denotes perfect information and $e = (1, \ldots, 1) \in S$ denotes *no information.*

Definition 5. Utility indices (v, u) *agree on the ranking* of lotteries $p, q \in \Delta X$ if either $[v(p) \geq v(q)$ and $u(p) \geq u(q)]$ or $[v(q) \geq v(p)$ and $u(q) \geq u(p)]$.

Definition 5 states that two utility indices agree on the ranking of p and q if both rank p weakly preferred to q or both rank q weakly preferred to p.

[14]There are many different presentations of Blackwell's characterization. See de Oliveira (2018), Bielinska-Kwapisz (2003), Crémer (1982), or Leshno and Spector (1992) for accessible treatments.

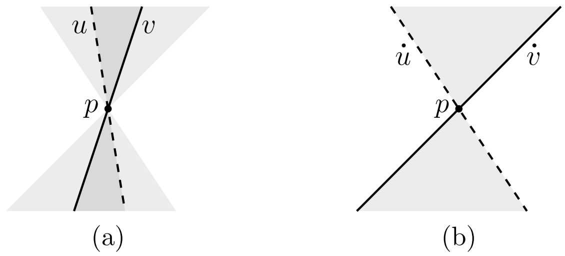


Figure 5: More-aligned utilities. In (a), v and u are nearly opposite preferences, as indicated by the narrow agreement region. In (b), the agreement region expands.

Definition 6. Utility indices $(\dot{v}, \dot{u})$ are *more aligned* than (v, u) if, for all $p, q \in \Delta X$, $(\dot{v}, \dot{u})$ agree on the ranking of p and q if (v, u) do.

Figure 5 illustrates Definition 6. The definition does not require v and $\dot{v}$ (or u and $\dot{u}$) to rank p and q the same way: v and u may rank p over q while $\dot{v}$ and $\dot{u}$ rank q over p. All that matters is that within each pair there is no disagreement regarding the ranking of p and q. If $\dot{v}$ and $\dot{u}$ are "between" v and u in that $\dot{v} = \alpha v + (1-\alpha)u$ and $\dot{u} = \beta v + (1-\beta)u$ for some $\alpha, \beta \in [0, 1]$, then $(\dot{v}, \dot{u})$ is more aligned than (v, u). Hence, the definition is not vacuous.

Proposition 1. *If (ν, μ, v, u) represents $\succsim$ and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$ represents $\dot{\succsim}$, then:*

(i) $(\dot{v}, \dot{u})$ is more aligned than (v, u) if and only if for all bets A, σ^* is top-ranked by $\dot{\succsim}^A$ if it is top-ranked by $\succsim^A$.

(ii) $v \approx u$ if and only if for all bets A, σ^* is top-ranked by $\succsim^A$.

Part (i) of Proposition 1 states that Sender and Receiver utilities are more aligned if there are more bets where perfect information is Sender-optimal. This holds because σ^* is Sender-optimal at a pq-bet if and only if (v, u) agree on the ranking of p and q. Part (ii) expresses the limit case $v \approx u$ where σ^* is Sender-optimal in all bets.

Definition 7. Let $E, F \subseteq \Omega$ and $s \in S$. Then (ν, μ) *agree on the ranking of E and F at s* if either $[\nu^s(E) \geq \nu^s(F)$ and $\mu^s(E) \geq \mu^s(F)]$ or $[\nu^s(F) \geq \nu^s(E)$ and $\mu^s(F) \geq \mu^s(E)]$, where ν^s and μ^s denote the Bayesian posteriors of ν and μ at signal s.

This definition states that ν and μ agree on the ranking of E and F at s if and only if the Bayesian posteriors ν^s and μ^s rank E and F the same way: either both assign higher probability to E, or both assign higher probability to F.

Definition 8. Priors $(\dot{\nu}, \dot{u})$ are *more aligned* than (ν, μ) if, for all $E, F \subseteq \Omega$ and all $s \in S$, $(\dot{\nu}, \dot{\mu})$ agree on the ranking of E and F at s whenever (ν, μ) do.

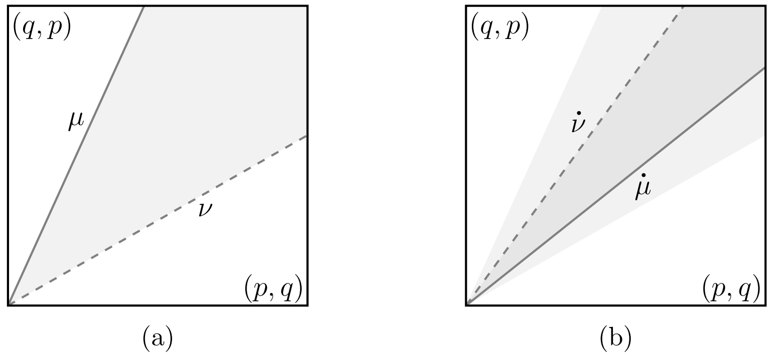


Figure 6: More-aligned priors. In (a), there is a relatively large gap between ν and μ—the slopes of the two lines are ν_1/ν_2 and μ_1/μ_2, respectively. In (b), the gap narrows.

Definition 8 states that Sender and Receiver priors are more aligned if, for any pair of events, there is a larger set of signals making the Bayesian posteriors agree on the ranking of those events. Thus, more-aligned priors make it "easier" for Sender and Receiver to agree on the ranking of events (see Figure 6). The definition is not vacuous: if $\alpha, \beta \in [0,1]$, $\dot{\nu} = \alpha\nu + (1-\alpha)\mu$, and $\dot{\mu} = \beta\nu + (1-\beta)\mu$, then $(\dot{\nu}, \dot{\mu})$ is more aligned than (ν, μ).

Preferences $\succsim^A$ are *Blackwell monotone on* $\mathcal{E}' \subseteq \mathcal{E}$ if either (i) $\sigma \succsim^A \sigma'$ for all $\sigma, \sigma' \in \mathcal{E}'$ where $\sigma \sqsupseteq \sigma'$, or (ii) $\sigma' \succsim^A \sigma$ for all $\sigma, \sigma' \in \mathcal{E}'$ where $\sigma \sqsupseteq \sigma'$. That is, $\succsim^A$ is Blackwell monotone on $\mathcal{E}'$ if it either *satisfies* (case (i)) or *reverses* (case (ii)) the relation $\sqsupseteq$ on $\mathcal{E}'$.

A bet A is *non-degenerate* if there exist $\sigma, \sigma' \in \mathcal{E}$ such that $\sigma \succ^A \sigma'$. An experiment σ is *EF-informative* if $\sigma \not\sim^A e$ for all non-degenerate EF-bets A; intuitively, EF-informative experiments create enough variation in Receiver's beliefs to impact Receiver's choices—and, thereby, Sender's value of information—in EF-bets.

Definition 9. An EF-informative experiment σ is *EF-extreme* if there exists a neighborhood $N^\varepsilon(\sigma)$ such that, for all non-degenerate EF-bets A, $\succsim^A$ satisfies the Blackwell ordering on $N^\varepsilon(\sigma)$ if σ^* is top-ranked by $\succsim^A$, and reverses the Blackwell ordering on $N^\varepsilon(\sigma)$ otherwise.

To understand Definition 9, observe that Sender and Receiver agree on the ranking of E and F at signals that are very informative, or "extreme." For example, a signal that perfectly reveals state ω yields a common posterior. Since Bayes' rule is continuous in s, any signal yielding a common strict ranking of E and F can be perturbed without reversing the ranking. So, any disagreement must occur at noisier signals, dividing S into "agreement" and "disagreement" regions (see Figure 6). The idea of the definition is that σ is EF-extreme if every $s \in \sigma$ belongs to the agreement region. To see this, consider a non-degenerate EF-bet $A = \{pEq, pFq\}$. Sender and Receiver either agree or disagree on the ranking of p and q. If they agree on the ranking, then they agree on the optimal act in A if they agree on which event (E or F) is more likely. Thus, near extreme experiments σ, Receiver's choices from A

coincide with what Sender would choose if, hypothetically, he were allowed to choose from A. Consequently, choice behavior coincides with that of a standard Bayesian and $\succsim^A$ satisfies the Blackwell ordering near σ. If instead Sender and Receiver disagree on the ranking of p and q, then Receiver's behavior near σ coincides with that of a standard Bayesian with utility index $-v$, making Sender's preferences reverse the Blackwell ordering near σ.

Proposition 2. *If (ν, μ, v, u) represents $\succsim$ and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$ represents $\dot{\succsim}$, then:*

(i) $(\dot{\nu}, \dot{\mu})$ is more aligned than (ν, μ) if and only if every EF-extreme experiment is $\dot{EF}$-extreme.

(ii) $\nu = \mu$ if and only if for all bets A, $\succsim^A$ is Blackwell monotone on $\mathcal{E}$.

Part (i) of Proposition 2 captures the idea illustrated by Figure 6 and formalized by the concept of extreme experiments. When more experiments are EF-extreme, more signals generate agreement between Sender and Receiver regarding the ranking of E and F. Hence, Sender's preferences are Blackwell monotone around a larger set of experiments in EF-bets. Part (ii) expresses the limit case where Sender and Receiver share a common prior. With a common prior, every signal yields a common posterior. Thus, in any pq-bet A, Sender's preferences either satisfy the Blackwell ordering on $\mathcal{E}$ or reverse it on $\mathcal{E}$, corresponding to whether (v, u) agree or disagree on the ranking of p and q. This logic only applies to bets: if A is not a bet, $\succsim^A$ need not satisfy or reverse the Blackwell ordering on $\mathcal{E}$ even with a common prior. For instance, Sender's preferences in the example from Section 1 (utilizing a common prior) neither satisfy nor reverse the Blackwell ordering on $\mathcal{E}$ because the menu in that example cannot be expressed as a bet.

Proposition 3. *If (ν, μ, v, u) represents $\succsim$, then $\nu = \mu$ and $v \approx u$ if and only if either of the following (equivalent) conditions hold:*

(i) For all menus A, $\sigma \sqsupseteq \sigma'$ implies $\sigma \succsim^A \sigma'$.

(ii) For all bets A, $\sigma \sqsupseteq \sigma'$ implies $\sigma \succsim^A \sigma'$.

Proposition 3 provides a characterization of the joint limit case where Sender and Receiver share a common prior and a common utility function (I omit the straightforward proof). In particular, $\nu = \mu$ and $v \approx u$ if and only if Sender's preferences $\succsim^A$ satisfy the Blackwell ordering in all menus A. Part (ii) establishes that, in fact, adherence to the Blackwell ordering in bets will suffice. Thus, in the behavioral interpretation of the model, the individual is dynamically consistent if and only if each preference $\succsim^A$ satisfies the Blackwell ordering.

6 Conclusion

Leveraging both Sender's preferences for information and Receiver's signal-contingent choices, this paper has characterized the testable implications of a large class of communication models with sender commitment power (Bayesian persuasion). An intermediate result characterizes Receiver as a Bayesian information processor, providing a novel foundation for such behavior. Sender and Receiver can be interpreted as a single individual, reflecting the behavior of a dynamically inconsistent decision maker who—lacking hard commitment power—influences future choice through selective exposure to information.

The results highlight the power of information structures (Blackwell experiments) as objects of choice. Although information is of purely instrumental value to Bayesian decision makers, Sender's preferences for information (conditioned on choice sets) fully reveal the priors and utility functions of both agents. Testable conditions on Sender's preferences also characterize the degree of separation between the beliefs or utilities of the two agents.

An advantage of the informational-preference approach is that it characterizes the interaction in terms of the choices agents actually make in disclosure models. Moreover, people frequently compare and choose information structures in daily life. The results of this paper demonstrate how observation of such choices might be used to test models and identify parameters, expanding the types of data that can be used in revealed-preference analysis.

A Proof of Theorem 1

Preliminaries

This section reviews some basic material on affine spaces. The *affine hull* of $Y \subseteq \mathbb{R}^n$ is $\text{aff}(Y) := \{\alpha^0 x^0 + \ldots + \alpha^m x^m : x^0, \ldots, x^m \in Y \text{ and } \sum_{i=0}^m \alpha^i = 1\}$. Elements of $\text{aff}(X)$ are *affine combinations* of X. Clearly, $\text{co}(Y) \subseteq \text{aff}(Y)$, where $\text{co}(Y)$ is the convex hull of Y.

A set $Y \subseteq \mathbb{R}^n$ is an *affine space* if $Y = \text{aff}(Y)$. Every affine space Y is of the form $Y = a + Z := \{a + z : z \in Z\}$ for some $a \in \mathbb{R}^n$ and linear subspace $Z \subseteq \mathbb{R}^n$. The *dimension* of an affine space $Y = a + Z$ is $\dim(Y) := \dim(Z)$. This definition extends to arbitrary convex sets $C \subseteq \mathbb{R}$ by letting $\dim(C) := \dim(\text{aff}(C))$.

The set ΔX can be identified with a convex subset of $\mathbb{R}^N$ (where $N = |X|$) and satisfies $\dim(\Delta X) = |X| - 1$. Similarly, F can be identified with the set $\Delta X \times \ldots \times \Delta X = (\Delta X)^{|\Omega|}$ and has dimension $|\Omega|(N-1)$. Finally, a convex subset $C \subseteq (\Delta X)^m$ ($m \geq 1$) has *full dimension* if $\dim(C) = \dim((\Delta X)^m)$; that is, if $(\Delta X)^m \subseteq \text{aff}(C)$.

A set $\{x^0, \ldots, x^m\} \subseteq \mathbb{R}^n$ is *affinely independent* if $\{x^1 - x^0, \ldots, x^m - x^0\}$ is linearly independent. If $Y \subseteq \mathbb{R}^n$ is an affine space of dimension m and $B = \{x^0, \ldots, x^m\} \subseteq Y$ is affinely independent, then B is an *affine basis* for Y. In that case, every $x \in X$ may be expressed in *affine coordinates*: for each $x \in X$, there are unique scalars $\alpha^0, \ldots, \alpha^m \in \mathbb{R}$ with $\sum \alpha^i = 1$ such that $x = \alpha^0 x^0 + \ldots \alpha^m x^m$.

If $C \subseteq \mathbb{R}^n$ is convex, then $T : C \to \mathbb{R}$ is *linear* if $T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y)$ whenever $x, y \in C$ and $\alpha \in [0,1]$; $T^* : C \to \mathbb{R}$ is *affine* if $T^*(\alpha^0 x^0 + \ldots \alpha^n x^n) = \alpha^0 T^*(x^0) + \ldots + \alpha^n T^*(x^n)$ whenever $x^i \in C$, $\alpha^0 x^0 + \ldots + \alpha^n x^n \in C$, and $\alpha^0 + \ldots + \alpha^n = 1$.

Step 1: Construction of candidate representation

Recall that $N = |X|$ and let u, μ denote Receiver's (non-constant) utility index and (full support) prior. This step of the proof constructs a menu A^* where the associated set of induced acts is rich enough to pin down candidates for ν and v.

For every $A \in \mathcal{A}$, let $F^A := \{c^\sigma(A) : \sigma \in \mathcal{E}^c(A)\}$; this is the set of induced acts for A. Observe that if $\sigma, \sigma' \in \mathcal{E}^c(A)$ and $\alpha \in (0,1)$, then $c^{\alpha\sigma \cup (1-\alpha)\sigma'}(A) = \alpha c^\sigma(A) + (1-\alpha)c^{\sigma'}(A)$. Thus, F^A is a convex subset of F. By Consistency, as well as Axioms 1.1–1.3, the restriction of $\succsim^A$ to $\mathcal{E}^c(A)$ translates into an ordering on F^A satisfying the standard Rationality, Independence, and Continuity axioms. Hence, by the Mixture Space Theorem of Herstein and Milnor (1953), there is a linear function $W^A : F^A \to \mathbb{R}$ such that for all $\sigma, \sigma' \in \mathcal{E}^c(A)$, $\sigma \succsim^A \sigma'$ if and only if $W^A(c^\sigma(A)) \geq W^A(c^{\sigma'}(A))$.

Lemma 1. *There exists an affinely independent set $P = \{p^1, \ldots, p^N\}$ of interior lotteries such that (i) $u(p^N) > u(p^{N-1}) > \ldots > u(p^1)$ and (ii) $u(p^2) - u(p^1) > u(p^3) - u(p^2) > \ldots > u(p^N) - u(p^{N-1})$.*

Proof. It is easy to find interior lotteries satisfying (i) and (ii). If necessary, perturb them along indifference curves (hyperplanes) in ΔX to arrive at an affinely independent set. □

For the remainder of Step 1, let P satisfy the requirements of Lemma 1. The set $\text{co}(P)$ has dimension $N-1$ by affine independence. Interpreting $\text{co}(P)$ as a polytope, each p^i is a vertex, and every nonempty $P' \subseteq P$ corresponds to a face $\text{co}(P')$ of dimension $|P'| - 1$.

Lemma 2. *Suppose $P' \subseteq P$, $D \subseteq co(P')$ is convex, and $\dim D = \dim co(P')$. If $\hat{p} \in P \backslash P'$ and $q^1, \ldots, q^n \in co(P' \cup \{\hat{p}\}) \backslash co(P')$, then $\dim \bigcap_{i=1}^n co(D \cup \{q^i\}) = \dim D + 1$.*

Proof. First, we prove the following claim: if $x \in D$, $q \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$, and $\varepsilon > 0$, then $\dim(\text{co}(D \cup \{q\}) \cap N^\varepsilon(x)) = \dim D + 1$, where $N^\varepsilon(x)$ is the ε-neighborhood of x.

To prove the claim, note that $\dim(\text{co}(D \cup \{q\})) = \dim D + 1$. Therefore, there exist $z^1, \ldots, z^K \in \text{co}(D \cup \{q\}) \backslash \{x\}$ ($K = \dim D + 1$) such that $\{x, z^1, \ldots, z^K\}$ is affinely independent. Thus, $\{z^1 - x, \ldots, z^K - x\}$ is linearly independent. For every $i = 1, \ldots, K$, the line L^i through x and z^i passes through $N^\varepsilon(x)$. For each i, let $x^i \in N^\varepsilon(x)\backslash\{x\}$ be a point on L^i. Then, since $\{z^1 - x, \ldots, z^K - x\}$ is linearly independent, the set $\{x, x^1, \ldots, x^K\}$ is affinely independent. It follows that $\text{co}\{x, x^1, \ldots, x^K\} \subseteq N^\varepsilon(x)$ has dimension $\dim D + 1$, and therefore $N^\varepsilon(x) \cap \text{co}(D \cup \{q\})$ has dimension $\dim D + 1$. This proves the claim.

Now fix a point x in the (relative) interior of D and apply the claim to each $q = q^1, \ldots, q^n \in \text{co}(P' \cup \{\hat{p}\})\backslash\text{co}(P')$. Since x is in the interior of D, there exists $\varepsilon_i > 0$ $(i = 1, \ldots, n)$ such that $N^{\varepsilon_i}(x) \cap \text{co}(D \cup \{\hat{p}\}) = N^{\varepsilon_i}(x) \cap \text{co}(D \cup \{q^i\})$. Let ε denote the smallest such ε_i and choose a point y in the relative interior of $\text{co}(P' \cup \{\hat{p}\}) \cap N^\varepsilon(x)$. By the claim, each set $N^\varepsilon(x) \cap \text{co}(D \cup \{q^i\})$ has dimension $\dim D + 1$, and $y \in \text{co}(P' \cup \{\hat{p}\})\backslash\text{co}(P')$. Since $y \in \bigcap_{i=1}^n \text{co}(D \cup \{q^i\})$, it follows that $\dim \bigcap_{i=1}^n \text{co}(D \cup \{q^i\})$ has dimension $D + 1$. □

For an ordered pair $E = [\omega, \omega']$ (where $\omega \neq \omega'$), lotteries p, q, and an act h, let $(p, q)Eh$ denote the act f such that $f_\omega = p$, $f_{\omega'} = q$, and $f_{\hat{\omega}} = h_{\hat{\omega}}$ for all $\hat{\omega} \neq \omega, \omega'$. Similarly, if $\alpha, \beta \in [0, 1]$ and $t \in S$, then $(\alpha, \beta)Et$ denotes the profile r where $r_\omega = \alpha$, $r_{\omega'} = \beta$, and $r_{\hat{\omega}} = t_{\hat{\omega}}$ for all $\hat{\omega} \notin E$. To qualify as a signal, r must have a nonzero entry.

Definition 10 (Symmetric Menu)**.** Let $u(p) > u(\underline{p})$ for all $p \in P$. For each $E = [\omega, \omega']$, let $A^E := \{(p^i, p^{N-i+1})E\underline{p} : i = 1, \ldots, N\} = \{(p^1, p^N)E\underline{p}, (p^2, p^{N-1})E\underline{p}, \ldots, (p^N, p^1)E\underline{p}\}$. The *symmetric menu on* $(P, \underline{p})$ is given by $A^* := \bigcup_E A^E$.

For the remainder of Step 1, let A^* satisfy the requirements of Definition 10. For every $\hat{\omega}$, let $e^{\hat{\omega}}$ denote a signal assigning likelihood 1 to state $\hat{\omega}$ and 0 to all other states.

Definition 11. For $E = [\omega, \omega']$, where $\omega \neq \omega'$, let $S^E := \{s \in S : \hat{\omega} \notin E \Rightarrow s_{\hat{\omega}} = 0\}$ and $\mathcal{E}^E := \{\sigma \in \mathcal{E} : \forall s \in \sigma,$ either $s \in S^E$ or $s = \lambda e^{\hat{\omega}}$ for some $\lambda \in (0, 1]$ and $\hat{\omega} \in \Omega\}$.

If $s \in S^E$, then states outside of E are assigned likelihood 0 by s. An experiment $\sigma \in \mathcal{E}^E$ is composed of signals from S^E as well as (scalar multiples of) indicator signals $e^{\hat{\omega}}$ for each $\hat{\omega} \notin E$. Observe that S^E and $\mathcal{E}^E$ are convex and that $c^s(A^*) \subseteq A^E$ if $s \in S^E$.

Lemma 3. *For each $E = [\omega, \omega']$ and $f \in A^E$, there is an $s \in S^E$ such that $c^s(A^*) = f$.*

Proof. We have $c^s(A^*) \subseteq A^E$ whenever $s \in S^E$. Therefore, we only need to show that for each $f \in A^E$, there is a signal $s \in S^E$ such that $f \succsim^s g$ for all $g \in A^E$.

First, observe that if $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p}$ for some $s \in S^E$, then $(p^{i+1}, p^{N-(i+1)+1})E\underline{p} \succ^s (p^{i+2}, p^{N-(i+2)+1})E\underline{p}$. Similarly, $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i-1}, p^{N-(i-1)+1})E\underline{p}$ implies $(p^{i-1}, p^{N-(i-1)+1})E\underline{p} \succ^s (p^{i-2}, p^{N-(i-2)+1})E\underline{p}$. These properties follow from the Bayesian Representation for c and the fact that P satisfies the requirements of Lemma 1. Thus, for $1 < i < N$, we have $c^s(A^*) = (p^i, p^{N-i+1})E\underline{p}$ if and only if

$$(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p} \quad \text{and} \quad (p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i-1}, p^{N-(i-1)+1})E\underline{p}.$$

Since $s \in S^E$, it cannot be the case that both $s_\omega = 0$ and $s_{\omega'} = 0$. Suppose $s_{\omega'} > 0$. By the Bayesian Representation for c, the conditions above are equivalent to

$$\frac{\mu_{\omega'}}{\mu_\omega}\frac{u(p^{N-(i-1)+1}) - u(p^{N-i+1})}{u(p^i) - u(p^{i-1})} < \frac{s_\omega}{s_{\omega'}} < \frac{\mu_{\omega'}}{\mu_\omega}\frac{u(p^{N-i+1}) - u(p^{N-(i+1)+1})}{u(p^{i+1}) - u(p^i)}.$$

Since P satisfies the requirements of Lemma 1, this yields an interval of values for $\frac{s_\omega}{s_{\omega'}}$ such that $c^s(A^*) = (p^i, p^{N-i+1})E\underline{p}$. The case $s_\omega > 0$ is similar.

For $i = 1$ or $i = N$, observe that $s \in S^E$ satisfies $c^s(A^*) = (p^1, p^N)E\underline{p}$ if and only if $(p^1, p^N)E\underline{p} \succ^s (p^2, p^{N-1})E\underline{p}$ while $c^s(A^*) = (p^N, p^1)E\underline{p}$ if and only if $(p^N, p^1)E\underline{p} \succ^s (p^{N-1}, p^2)E\underline{p}$. Using the Bayesian Representation in a similar manner, it follows that there exist signals $s \in S^E$ such that $c^s(A^*) = (p^1, p^N)E\underline{p}$. □

Lemma 4. *There exists a convex, full-dimensional set $D \subseteq co(P)$ such that, for every $p \in D$ and every state ω, there is an experiment σ such that $c^\sigma_\omega(A^*) = p$.*

Proof. We will construct D in several steps. First, enumerate $\Omega = \{1, \ldots, W\}$. We will work with pairs of the form $E = [1, \omega]$ for $\omega = 2, \ldots, W$.

Consider $E = [1, \omega]$. Under perfect information σ^*, we have $c^{\sigma^*}_{\hat\omega}(A^*) = p^N$ for all $\hat\omega$. Notice that $\sigma^* \in \mathcal{E}^E$. There exists $\delta > 0$ such that for $s = (1-\delta, \delta)E0$ and $t = (\delta, 1-\delta)E0$, we have $c^s(A^*) = (p^N, p^1)E\underline{p}$ and $c^t(A^*) = (p^1, p^N)E\underline{p}$. Thus, the experiment $\sigma = [s, t] \cup [e^{\hat\omega} : \hat\omega \notin E]$ yields $c^\sigma(A^*) = (\delta p^1 + (1-\delta)p^N, \delta p^1 + (1-\delta)p^N)Ep^N$, so $c^\sigma_1(A^*) = c^\sigma_\omega(A^*) = \delta p^1 + (1-\delta)p^N$. Mixing σ^* with σ yields a convex set $D^1_\omega \subseteq \text{co}(\{p^1, p^N\})$ of dimension 1 such that, for all $p \in D^1_\omega$, there exists σ such that $c^\sigma_{\omega'}(A^*) = p$ for $\omega' = 1, \omega$. Since every such set lies on the face $\text{co}\{p^1, p^N\}$ and contains p^N, the set $D^1 := \bigcap_{\omega \geq 2} D^1_\omega$ is nonempty and $\dim(D^1) = 1$.

We now proceed by induction. Suppose $D^i \subseteq \text{co}\{p^1, \ldots, p^i, p^N\}$ $(1 \leq i < N)$ is a convex set of dimension i and, for all $p \in D^i$ and all ω, there exists $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$. We construct a convex set $D^{i+1} \subseteq \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\}$ of dimension $i+1$ such that, for every $p \in D^{i+1}$ and $E = [1, \omega]$, there exists $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$. Take $E = [1, \omega]$ and $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$ for some $p \in \text{int}D^i$. There exists $t \in \sigma$ such that $c^t(A^*)$ is a singleton and $t_{\hat\omega} > 0$ for $\hat\omega \in E$ (otherwise, $c^\sigma_\omega(A^*) = p^N \notin \text{int}D^i$). Let $s \in S^E$ such that $c^s(A^*) = (p^{N-i}, p^{i+1})E\underline{p}$. We may assume that $t - s \in S$ and $c^{t-s}(A^*) = c^t(A^*)$ (if necessary, replace s with λs for a sufficiently small λ). Let σ' be an experiment formed by deleting t from σ and appending $t - s$ and s. Then $q^\omega := c^{\sigma'}_\omega(A^*) \in \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\} \backslash \text{co}\{p^1, \ldots, p^i, p^N\}$. Repeating this for all ω (as well as $E = [2, \omega = 1]$) yields lotteries $q^1, \ldots, q^W \in \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\} \backslash \text{co}\{p^1, \ldots, p^i, p^N\}$. Taking mixtures of σ' and σ (and varying σ to generate $c^\sigma_\omega(A^*) = p$ for all $p \in \text{int}D^i$) implies every $q \in \text{co}\{\text{int}D^i, q^\omega\}$ satisfies $c^{\sigma''}_\omega(A^*) = q$ for some $\sigma'' \in \mathcal{E}^E$. By Lemma 2, $D^{i+1} := \bigcap_{\omega=1}^W \text{co}(\text{int}D^i \cup \{q^\omega\})$ has dimension $i+1$. By construction, for every $p \in D^{i+1}$ and every ω there is an experiment $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$. $\square$

For the remainder of Step 1, let $D \subseteq \text{co}(P)$ satisfy all requirements of Lemma 4.

Lemma 5. *Suppose $L^*_\omega \subseteq \Delta X$ is full-dimensional for all ω. Let $f^* \in F$ and $L^*_\omega[\omega]f^* := \{p[\omega]f^* : p \in L^*_\omega\}$. If $G \subseteq F$ is convex and $L^*_\omega[\omega]f^* \subseteq G$ for all ω, then G has full dimension.*

Proof. Observe that for each ω, $\text{aff}(G) \supseteq \text{aff}(L^*_\omega[\omega]f^*) = \{p[\omega]f^* : p \in \Delta X\}$ since L^*_ω has full dimension in ΔX. Therefore $\text{aff}(G) \supseteq \text{aff}(C)$, where $C = \bigcup_\omega \{p[\omega]f^* : p \in \Delta X\}$. Let $g \in F$ and $\alpha = \frac{1}{|\Omega|}$. Then $g_\omega[\omega]f^* \in C$ for all ω, so $h := \sum_\omega \alpha g_\omega[\omega]f^* = \alpha g + (1-\alpha)f^* \in \text{co}(C)$. Since h is on the line connecting f^* and g, it follows that $g \in \text{aff}(C)$. Thus, $\text{aff}(C) = F$. $\square$

Definition 12. Fix a menu A. For each $f \in A$, let $S^A(f) := \{s \in S : c^s(A) = f\}$. A set σ of signals (not necessarily qualifying as an experiment) is *A-interior* if (i) $c^s(A)$ is single-valued for all $s \in \sigma$, and (ii) for each $f \in A$, there is exactly one $s \in \sigma$ such that $c^s(A) = f$.

Let S^* denote the set of all signals s such that $s_\omega > 0$ for all ω. The statement $\sigma \subseteq S^*$ means σ is a matrix (not necessarily an experiment) where each column is a member of S^*.

Definition 13. Suppose $\sigma \subseteq S^*$ is A-interior and let $\varepsilon > 0$. For each $s \in \sigma$, let $Q^{s,\varepsilon} := \prod_\omega (s_\omega - \varepsilon, s_\omega + \varepsilon)$. Let B^ε denote the set of all A-interior matrices $\sigma' \subseteq S^*$ such that (i) for each ω, $\sum_{s' \in \sigma'} s'_\omega = \sum_{s \in \sigma} s_\omega$, and (ii) if $s \in \sigma$, $s' \in \sigma'$, and $c^s(A) = c^{s'}(A)$, then $s' \in Q^{s,\varepsilon}$. Then B^ε is an *ε-neighborhood* of σ in A.

Note that Definition 13 does not require σ to be an experiment, and that $B^\varepsilon \subseteq \mathcal{E}$ (in fact, $B^\varepsilon \subseteq \mathcal{E}^c(A)$) if and only if σ is an experiment.

Lemma 6. *Suppose $\sigma \in \mathcal{E}$ is A-interior and that for each ω, there exists $B \subseteq A$ where $|B| = N$ and $B_\omega := \{f_\omega : f \in B\}$ is affinely independent. If B^ε is an ε-neighborhood for σ, then:*

(i) For each ω, $F^A(B^\varepsilon) := \{c^{\sigma'}(A) : \sigma' \in B^\varepsilon\}$ has a subset of the form $\{p[\omega]f^* : p \in L^*\}$, where $L^* \subseteq \Delta X$ is full-dimensional and $f^* = c^\sigma(A)$; and

(ii) $F^A(B^\varepsilon)$ contains a full-dimensional ball around $c^\sigma(A)$.

Proof. For (i), fix ω and let $f^* = c^\sigma(A)$ and $f^*_{-B} := \sum_{s \in \sigma^{-B}} s_\omega c^s_\omega(A)$, where $\sigma^B := \{s \in \sigma : c^s(A) \in B\}$ and $\sigma^{-B} := \sigma \backslash \sigma^B$. Then $|\sigma^B| = N$. Without loss of generality, let B^ε denote an ε-neighborhood of σ^B. For every $\sigma' \in B^\varepsilon$, there is a natural bijection between signals of σ and signals of σ'; specifically, $s \in \sigma$ and $s' \in \sigma'$ are related if and only if $c^s(A) = f^s = c^{s'}(A)$. For each $s \in \sigma$, let s' denote the corresponding signal in σ'.

Consider $\sigma' \in B^\varepsilon$ such that for all $s \in \sigma$ and all $\omega' \neq \omega$, $s_{\omega'} = s'_{\omega'}$. Every such σ' induces an act of the form $p[\omega]f^*$, where

$$\begin{aligned} p &\in \left\{ \sum_{s' \in \sigma'} s'_\omega f^s_\omega + f^*_{-B} : |s'_\omega - s_\omega| < \varepsilon \text{ for all } s' \in \sigma' \text{ and } \sum_{s' \in \sigma'} s'_\omega = \sum_{s \in \sigma^B} s_\omega \right\} \\ &= \left\{ \sum_{s \in \sigma^B} (s_\omega + \delta^s) f^s_\omega + f^*_{-B} : |\delta^s| < \varepsilon \text{ and } \sum_{s \in \sigma^B} \delta^s = 0 \right\} \\ &= \left\{ f^*_\omega + \sum_{s \in \sigma^B} \delta^s f^s_\omega : |\delta^s| < \varepsilon \text{ and } \sum_{s \in \sigma^B} \delta^s = 0 \right\}. \end{aligned}$$

So, it will suffice to show that the set $C := \left\{ \sum_{s \in \sigma^B} \delta^s f^s_\omega : |\delta^s| < \varepsilon \text{ and } \sum_{s \in \sigma^B} \delta^s = 0 \right\}$ has dimension $N-1$ (clearly, C is convex). Note that $N-1$ is an upper bound on the dimension of C because C is a translation of a subset of ΔX. Pick any $s^* \in \sigma^B$ and note that the set $C' := \{\frac{\varepsilon}{2} f^s_\omega - \frac{\varepsilon}{2} f^{s^*}_\omega : s \in \sigma^B \backslash s^*\} \subseteq C$ has cardinality $N-1$ and is linearly independent because $B_\omega = \{f^s_\omega : s \in \sigma^B\}$ is affinely independent. Thus, $\{0\} \cup C' \subseteq C$ is affinely independent, so that $\dim(C) = N-1$.

For (ii), note that by part (i), $F^A(B^\varepsilon)$ (hence F^A) contains a subset of the form $L^*_\omega[\omega]f^*$ for each ω, where each set $L^*_\omega \subseteq \Delta X$ has full dimension. Now apply Lemma 5. □

Lemma 7. *There is a full-dimensional set $L^* \subseteq \Delta X$ such that, for all ω, there exists $h^\omega \in F$ such that $L^*[\omega]h^\omega := \{p[\omega]h^\omega : p \in L^*\} \subseteq F^{A^*}$.*

Proof. Choose a lottery p^* in the interior of D and fix ω. Then there is an A^*-interior experiment σ such that $c^\sigma(A^*) = p^*[\omega]h$ for some $h \in F$. By part (ii) of Lemma 6, F^{A^*} contains a full-dimensional ball around $p^*[\omega]h$. In particular, there is a convex, full-dimensional set $L^*_\omega \subseteq \Delta X$ such that p^* belongs to the interior of L^*_ω and $\{p[\omega]h : p \in L^*_\omega\} \subseteq F^{A^*}$. We may assume that $L^*_\omega \subseteq D$. Since $p^* \in D$, we can repeat this argument for all ω to get a family of convex, full-dimensional sets $L^*_\omega \subseteq \Delta X$, each containing p^* as an interior point, and acts $h^\omega \in F$ such that $\{p[\omega]h^\omega : p \in L^*_\omega\} \subseteq F^{A^*}$. Letting $L^* := \bigcap_{\omega \in \Omega} L^*_\omega$ completes the proof. □

Lemma 8. *Any linear representation $W^{A^*} : F^{A^*} \to \mathbb{R}$ of $\succsim^{A^*}$ on $\mathcal{E}^c(A^*)$ has a unique linear extension $W : F \to \mathbb{R}$. The extension represents a preference $\succsim$ on F satisfying all of the Anscombe-Aumann axioms except (possibly) the Non-Degeneracy axiom.*

Proof. As explained at the start of Step 1, a linear representation W^{A^*} exists. By Lemmas 5 and 7, F^{A^*} has full dimension, and therefore W^{A^*} has a unique linear extension $W : F \to \mathbb{R}$. This induces a complete and transitive relation $\succsim$ on F by letting $f \succsim g$ if and only if $W(f) \geq W(g)$. The Independence and Continuity axioms are satisfied by linearity of W.

To verify that $\succsim$ satisfies the State Independence axiom, suppose $p[\omega]h \succsim q[\omega]h$ and let $\omega' \in \Omega$ and $h' \in F$. We want to show that $p[\omega']h' \succsim q[\omega']h'$. By a standard result, there exist linear functions $U_\omega : \Delta X \to \mathbb{R}$ (unique up to positive affine transformation) such that $W(f) = \sum_\omega U_\omega(f_\omega)$ for all $f \in F$. Thus, $p[\omega]h \succsim q[\omega]h$ implies $U_\omega(p) \geq U_\omega(q)$.

Since $L^*[\omega]h^\omega \subseteq F^{A^*}$ for each ω, where $L^* \subseteq \Delta X$ is convex and has full dimension, there exists $r \in L^*$ and $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha)r \in L^*$ and $\alpha q + (1-\alpha)r \in L^*$. Thus, $(\alpha p+(1-\alpha)r)[\omega]h^\omega$, $(\alpha q+(1-\alpha)r)[\omega]h^\omega$, $(\alpha p+(1-\alpha)r)[\omega']h^{\omega'}$, and $(\alpha q+(1-\alpha)r)[\omega']h^{\omega'}$ are elements of F^{A^*}. Moreover, $(\alpha p + (1-\alpha)r)[\omega]h^\omega \succsim (\alpha q + (1-\alpha)r)[\omega]h^\omega$ because $W((\alpha p+(1-\alpha)r)[\omega]h^\omega) \geq W((\alpha q+(1-\alpha)r)[\omega]h^\omega)$ if and only if $U_\omega(p) \geq U_\omega(q)$.

Since $\succsim^{A^*}$ satisfies State Independence (Axiom 1.6) on the domain $\mathcal{E}^c(A^*)$, it follows that $(\alpha p + (1-\alpha)r)[\omega']h^\omega \succsim (\alpha q + (1-\alpha)r)[\omega']h^\omega$. Therefore $U_{\omega'}(p) \geq U_{\omega'}(q)$, so that $U_{\omega'}(p) + \sum_{\hat\omega\neq\omega} U_{\hat\omega}(h'_{\hat\omega}) \geq U_{\omega'}(q) + \sum_{\hat\omega\neq\omega} U_{\hat\omega}(h'_{\hat\omega})$. Thus, $p[\omega']h' \succsim q[\omega']h'$, as desired. □

Step 2: Spreading the representation

For the remainder of the proof, assume u has been normalized to take values in $[0,1]$. A binary relation $\succsim^*$ on F is a *linear preference relation* if it has a linear representation.

Definition 14. Let A and B be menus such that $\mathcal{E}^c(A)$ and $\mathcal{E}^c(B)$ are nonempty.

(i) A relation $\succsim^*$ on F *agrees with* $\succsim^A$ if, for all $\sigma, \sigma' \in \mathcal{E}^c(A)$, $\sigma \succsim^A \sigma' \Leftrightarrow c^\sigma(A) \succsim^* c^{\sigma'}(A)$.

(ii) A *inherits* a representation from B if every linear preference relation $\succsim^*$ on F that agrees with $\succsim^B$ also agrees with $\succsim^A$.

(iii) A and B *share a representation* if there is a unique linear preference relation $\succsim^*$ on F that agrees with both $\succsim^A$ and $\succsim^B$.

Lemma 9. *Let A and B be menus such that $\mathcal{E}^c(A)$ and $\mathcal{E}^c(B)$ are nonempty.*

(i) If $\dim(F^A) = \dim(F^A \cap F^B) \leq \dim(F^B)$, then A inherits a representation from B.

(ii) If $\dim(F^A \cap F^B) = \dim(F)$, then A and B share a representation.

Proof. By the Consistency axiom, $\succsim^A$ and $\succsim^B$ agree on the domain $F^A \cap F^B$. The restriction of W^B to $F^A \cap F^B$ is a linear function L. Since $F^A \cap F^B$ is convex and $\dim(F^A) = \dim(F^A \cap F^B) \leq \dim(F^B)$, L has a linear extension to F^A. Every such extension represents a linear preference relation $\succsim^*$ on F^A that agrees with $\succsim^A$ and $\succsim^B$, proving (i). For (ii), note that L has a unique linear extension to F whenever $\dim(F^A \cap F^B) = \dim(F)$. □

Definition 15. Let A be a menu.

1. If $f \in A$, the *A-support* of f is the set $S^A(f) := \{s \in S : c^s(A) = f\}$.

2. A is a *k-menu* if $|A| = k \geq 2$ and each $f \in A$ has nonempty A-support.

3. A is *independent* if it is a k-menu for some k and, for each ω, there is an N-menu $B \subseteq A$ such that $B_\omega := \{f_\omega : f \in B\}$ is affinely independent.

Lemma 10. *Suppose A is a k-menu.*

(i) If $f \in A$, then $S^A(f)$ is a convex cone and has full dimension (in S).

(ii) There exists an A-interior experiment σ.

(iii) If A is independent, then F^A has full dimension (in F).

Proof. For (i), observe that $s \in S^A(f)$ if and only if, for all $g \in A\backslash\{f\}$, $\sum_\omega s_\omega \mu_\omega u(f_\omega) > \sum_\omega s_\omega \mu_\omega u(g_\omega)$. Thus, if $s, t \in S^A(f)$, then $\lambda s \in S^A(f)$ for all $\lambda > 0$ and $\alpha s + (1-\alpha)t \in S^A(f)$ for all $\alpha \in [0,1]$, so that $S^A(f)$ is a convex cone. Since the inequality above is strict, there is an open ball around each $s \in S^A(f)$ preserving the inequality; thus, $S^A(f)$ is a full-dimensional subset of S.

For (ii), note that since A is finite and $S^A(f)$ is a convex cone, there are signals s^f $(f \in A)$ such that $c^{s^f}(A) = f$ and, for each ω, $\sum_{f \in A} s^f_\omega \leq 1$. For each ω, there is an $f \in A$ such that $u(f_\omega) \geq u(g_\omega)$ for all $g \in A$. Thus, s^f_ω can be increased as needed to ensure $\sum_{f \in A} s^f_\omega = 1$. Repeat this for each ω to get a well-defined experiment $\sigma = \{s^f : f \in A\}$.

For (iii), invoke part (ii) to get an A-interior σ and, hence, a ε-neighborhood around σ. Let $\omega \in \Omega$. Since A is independent, there is an N-menu $B \subseteq A$ such that $B_\omega = \{f_\omega : f \in B\}$ is affinely independent. Now apply Lemma 6. □

Definition 16. A finite, nonempty set $\mathcal{C}$ of convex cones in S is a *conic decomposition* if $\mathcal{C} = \{S^A(f) : f \in A\}$ for some k-menu A. For each k-menu A, the set $\mathcal{C}(A) := \{S^A(f) : f \in A\}$ is the *conic decomposition for A*.

Definition 17. For each k-menu A and $f \in A$, let $U(f) := (\mu_\omega u(f_\omega))_{\omega \in \Omega}$ denote the *(virtual) utility coordinate* for f, and let $U(A) := \{U(f) : f \in A\}$ denote the *utility profile for A*. If a set $U \subseteq \mathbb{R}^\Omega_+$ satisfies $U = U(A)$ for some k-menu A, then U is a *k-utility profile*. Finally, a finite set $U \subseteq \mathbb{R}^\Omega_+$ is a *utility profile* if U is a k-utility profile for some k.

Lemma 11. *If A and B are k-menus such that $U(A) = U(B)$, then $\mathcal{C}(A) = \mathcal{C}(B)$.*

Proof. This follows immediately from the definition of $U(A)$ and the fact that $s \in S^A(f)$ if and only if $\sum_\omega s_\omega \mu_\omega u(f_\omega) > \sum_\omega s_\omega \mu_\omega u(g_\omega)$ for all $g \in A\backslash\{f\}$. □

By Lemma 11, each utility profile U has an associated conic decomposition $\mathcal{C}(U)$. Specifically, $\mathcal{C}(U)$ is the unique $\mathcal{C}$ such that $U(A) = U$ implies $\mathcal{C}(A) = \mathcal{C}$.

Definition 18. Let U be a utility profile and $z = (z_\omega)_{\omega\in\Omega} \in U$. The *support* of z in U is the set $S^U(z) := \{s \in S : \forall z' \in U\backslash\{z\},\ \sum_\omega s_\omega z_\omega > \sum_\omega s_\omega z'_\omega\}$.

Definition 19. Let U be a utility profile. For each $z \in U$ and $s \in S^U(z)$, let $H(z,s) := \{\lambda \in \mathbb{R}^\Omega : s \cdot (\lambda - z) \leq 0\}$. The *support polytope of z in U*, denoted $T(z,U)$, is defined as $T(z,U) := \bigcap_{s\in S^U(z)} H(z,s)$. The *polytope of U*, denoted $T(U)$, is given by $T(U) := \bigcap_{z\in U} T(z,U)$. A polytope $T \subseteq \mathbb{R}^\Omega$ is a *decision polytope* if $T = T(U)$ for some utility profile U; it is a *k-polytope* if $T = T(U(A))$ for some k-menu A.

Definition 20. Let T be a decision polytope. For each face F of T, let $\eta^F \in S^\Omega_+ := \{\eta \in \mathbb{R}^\Omega_+ : \|\eta\| = 1\}$ such that η^F is normal to the hyperplane associated with F. Let $\mathcal{N}(T) := \{\eta^F : F \text{ is a face of } T\}$ denote the set of *normals* for T.

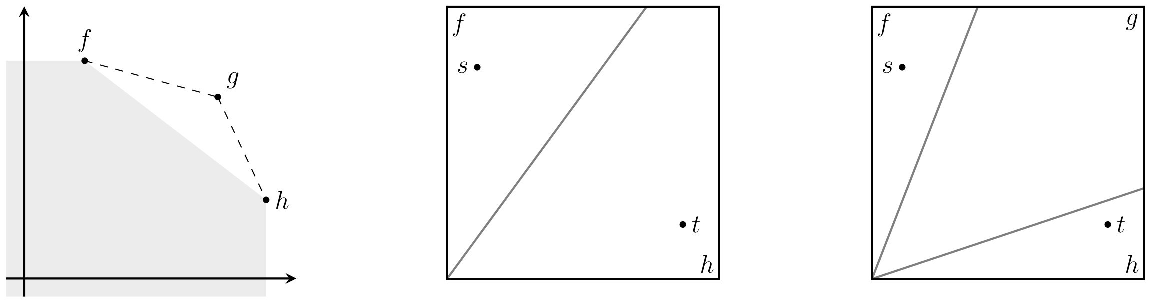
Figure 1 in the main text illustrates the relationship between a menu A, its utility profile $U(A)$, and the associated decision polytope and conic decomposition. In the figure, the shaded region is $T(U(A))$. An act is chosen under some signal if and only if $U(f)$ is an extreme point of the polytope. For any such act f, the set of signals s where $c^s(A) = f$ is a cone in S. Faces of the polytope correspond to signals making Receiver indifferent between two or more acts in A. Thus, any s perpendicular to a face of the polytope lies on a hyperplane in signal space separating the cones corresponding to two or more acts.

Lemma 12 (Vertex Expansion)**.** *Let A be a k-menu. There is an act $g \notin A$ such that $B = A \cup \{g\}$ is a $(k+1)$-menu and A inherits a representation from B.*

Proof. Let $\sigma \in \mathcal{E}$ be A-interior and choose $2\varepsilon > 0$ such that $B^{2\varepsilon}$ is a 2ε-neighborhood of σ. Then B^ε is an ε-neighborhood where, for all $\sigma' \in B^\varepsilon$ and all $s \in \sigma'$, the closure of $Q^{s,\varepsilon}$ is in the interior of $S^A(f)$, where $f = c^s(A)$.

Let $f \in A$. For each $\sigma' \in B^\varepsilon$ and each $s \in \sigma'$, consider the half-space $H(f,s) := \{\lambda \in \mathbb{R}^\Omega_+ : s \cdot (\lambda - U(f)) \leq 0\}$. This is the half-space (containing the origin) where the bounding hyperplane has normal s and passes through $U(f)$. Let T^* be (the closure of) the intersection over all $H(f,s)$ where $f \in A$ and $s \in \sigma' \in B^\varepsilon$. Notice that for each f, the set $B^\varepsilon(f) := \{s \in S : c^s(A) = f \text{ and } s \in \sigma' \in B^\varepsilon\}$ is an (open) convex cone in S, and a strict subset of $\text{int}(S^A(f))$ by our choice of ε. Thus, $B^\varepsilon(f)$ and $B^\varepsilon(f')$ are strictly separated whenever $f \neq f'$, and therefore $T(A) \subsetneq T^*$. Pick any point $u^* \in [T^*\backslash T(A)] \cap \mathbb{R}^\Omega_+$ and let $g \in F$ such that $U(g) = u^*$. Then $B = A \cup \{g\}$ is the desired menu.

To see why A and B share a representation, note that (by construction) $c^s(A) = c^s(B)$ for all $s \in \sigma' \in B^\varepsilon$. Hence, $c^{\sigma'}(A) = c^{\sigma'}(B)$ whenever $\sigma' \in B^\varepsilon$. Since $\dim(F^A) = \dim(F^A(B^\varepsilon))$ and $F^A(B^\varepsilon) = F^B(B^\varepsilon) \subseteq F^B$, it follows that $\succsim^A$ inherits a representation from $\succsim^B$. $\square$



(a) Utility Space

(b) Signal space for $A = \{f, h\}$

(c) Signal space for $B = \{f, g, h\}$

Figure 7: Illustration of Lemma 12. The shaded region in (a) is $T(A)$. Experiment $\sigma = [s, t]$ is constructed so that f is chosen at s and h is chosen at t. If the (utility coordinate) of g is close to the face joining f and h, then $c^s(B) = f$ and $c^t(B) = h$ as well, where $B = \{f, g, h\}$. How close g needs to be to the face depends on s and t (the dashed lines in (a) are perpendicular to the gray lines in (c)). Thus, $c^{\sigma'}(A) = c^{\sigma'}(B)$ for all σ' in a neighborhood of σ, so that A inherits a representation from B.

Lemma 13. *Let A be a k-menu. There exists an independent menu B such that A inherits a representation from B.*

Proof. Fix an A-interior experiment σ and a neighborhood B^ε of the form used in the proof of Lemma 12. It is easy to see that a similar argument can be used to add N additional vertices to the region $T^*\backslash T(A)$ to yield a $(k+N)$-polytope. Moreover, these vertices can be chosen so that for each state ω, the ω coordinates yield N distinct, interior utility values. Pick any N lotteries $p^1_\omega, \ldots, p^N_\omega$ yielding these utility values; these can be chosen to form an affinely independent set. Now let $f^i = (p^i_\omega)_{\omega\in\Omega} \in F$, and let $B = A \cup \{f^1, \ldots, f^N\}$. □

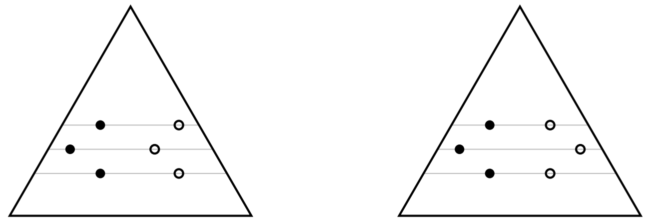
Definition 21. Let A and B be independent menus. Then B is a *translation* of A if there exists $\lambda^* \in \mathbb{R}^\Omega$ such that $T(B) = T(A) + \lambda^* := \{\lambda + \lambda^* : \lambda \in T(A)\}$. The notation $B = A + \lambda^*$ means $T(B) = T(A) + \lambda^*$.

Lemma 14. *If $B = A + \lambda^*$, then:*

(i) The map $\psi : U(A) \to U(B)$ given by $\psi(z) := z + \lambda^*$ is a bijection. Hence, there is a bijection $\psi : A \to B$ where $\psi(f)$ denotes the unique $g \in B$ such that $U(g) = U(f) + \lambda^*$.

(ii) $\mathcal{C}(B) = \mathcal{C}(A)$.

Proof. Part (i) is clear. For part (ii), observe that $s \in S^A(f)$ if and only if, for all $g \in A\backslash\{f\}$, $\sum_\omega s_\omega u(f_\omega) > \sum_\omega s_\omega u(g_\omega) \Leftrightarrow \sum_\omega s_\omega[\mu_\omega u(f_\omega) + \lambda^*_\omega] > \sum_\omega s_\omega[\mu_\omega u(g_\omega) + \lambda^*_\omega] \Leftrightarrow s \in S^B(\psi(f))$. It follows that $\mathcal{C}(B) = \mathcal{C}(A)$. □



(a) Oriented (b) Not Oriented

Figure 8: Orientedness when $|X| = 3$. The lotteries $A_\omega = \{f_\omega, g_\omega, h_\omega\}$ are solid dots and the lotteries $B_\omega = \{f'_\omega, g'_\omega, h'_\omega\}$ are circles. In this case, $\lambda = 0$ (the lines are indifference curves for u). The configuration in (b) is not oriented because the affine path from A_ω to B_ω (traversing along the lines) yields a collinear set of lotteries at $\alpha = 1/2$.

Definition 22. Let B be a translation of A and $\psi : A \to B$ denote the associated bijection (Lemma 14). The *affine path from f to $\psi(f)$* is the map $\alpha \mapsto f^\alpha := (1-\alpha)f + \alpha\psi(f)$ for $\alpha \in [0,1]$ and the *affine path from A to B* is the map $\alpha \mapsto A^\alpha := \{f^\alpha : f \in A\}$ for $\alpha \in [0,1]$.

Definition 23. A bijection $\varphi : P \to Q$ between to sets of N lotteries is *oriented* if (i) for all $p, p' \in P$, $u(p) > u(p')$ implies $u(\varphi(p)) > u(\varphi(p'))$, and (ii) for all $\alpha \in [0,1]$, the set $\{(1-\alpha)p + \alpha\varphi(p) : p \in P\}$ is affinely independent. Independent menus A and B are *oriented* if B is a translation of A and, for each ω, the map $\varphi_\omega : A_\omega \to B_\omega$ given by $\varphi_\omega(f_\omega) := \psi(f)_\omega$ is oriented, where $A_\omega := \{f_\omega : f \in A\}$, $B_\omega := \{g_\omega : g \in B\}$, and $\psi : A \to B$ is the associated bijection (Lemma 14).

Figure 8 illustrates the concept of orientedness. Note that not all translations $B = A + \lambda^*$ are oriented; as the figure shows, it is possible to construct menus A and B such that $U(A) = U(B)$ (so that B is trivially a translation of A) but where A and B are not oriented.

Lemma 15. *If A and B are oriented menus, then A and B share a representation.*

Proof. Since A and B are oriented, there is a $\lambda^* \in \mathbb{R}^\Omega$ such that $B = A + \lambda^*$ and an associated bijection $\psi : A \to B$ (Lemma 14). Consider the affine path associated with ψ (Definition 22), and note that for each α, $A^\alpha = A + \alpha\lambda^*$; that is, $T(A^*) = T(A) + \alpha\lambda^*$. Thus, every A-interior (B-interior) experiment σ is also A^α-interior. Pick such a σ and a corresponding neighborhood B^ε, and let $f^\alpha := c^\sigma(A^\alpha)$. Importantly, $F^{A^\alpha}(B^\varepsilon)$ contains a full-dimensional subset of F because A^α is an independent menu (A and B are oriented). Every $c^{\sigma'}(A^\alpha)$ ($\sigma' \in B^\varepsilon$) is of the form $c^{\sigma'}(A^\alpha) = f^\alpha + \sum_{s \in \sigma} \delta^s[(1-\alpha)f^s + \alpha\psi(f^s)]$, where $\sum_{s\in\sigma} \delta^s = 0$, $|\delta^s_\omega| < \varepsilon$, and $f^s = c^s(A)$. Thus, f^α is in the interior of $F^{A^\alpha}(B^\varepsilon)$ and there is a scalar $\delta^* > 0$ such that, for every α, $F^{A^\alpha}(B^\varepsilon)$ contains an open ball of radius δ^* around f^α.

Now construct a finite sequence $\alpha(0), \alpha(1), \ldots, \alpha(I)$ such that $\alpha(0) = 0$, $\alpha(I) = 1$, and $d(f^{\alpha(i)}, f^{\alpha(i-1)}) < \delta^*/2$ for all $i = 1, \ldots, I$, where d denotes the standard Euclidean metric.

This can be done because f^α is continuous in α. Notice that $f^{\alpha(i)} \in B^{\alpha(i-1)}$ for all $i = 1, \ldots, I$. Thus, $B^{\alpha(i)} \cap B^{\alpha(i-1)}$ has full dimension, so that $A^{\alpha(i)}$ and $A^{\alpha(i-1)}$ share a representation. Hence, A and B share a representation. $\square$

Lemma 16 (Face Expansion)**.** *Let A be an independent menu and $\lambda \in S_+^\Omega$. There is an independent menu B such that $\mathcal{N}(B) = \mathcal{N}(A) \cup \{\lambda\}$ and A and B share a representation.*

Proof. Fix an A-interior experiment σ and an ε-neighborhood B^ε around σ. Without loss of generality, no $s \in \sigma$ is of the form $s = \gamma\lambda$ for any $\gamma > 0$ (if necessary, choose some other $\sigma' \in B^\varepsilon$ and redefine σ to be σ'). Let $f^* := c^\sigma(A)$. Since A is independent, the set $F^A(B^\varepsilon)$ contains a ball of radius δ around f^* for some $\delta > 0$.

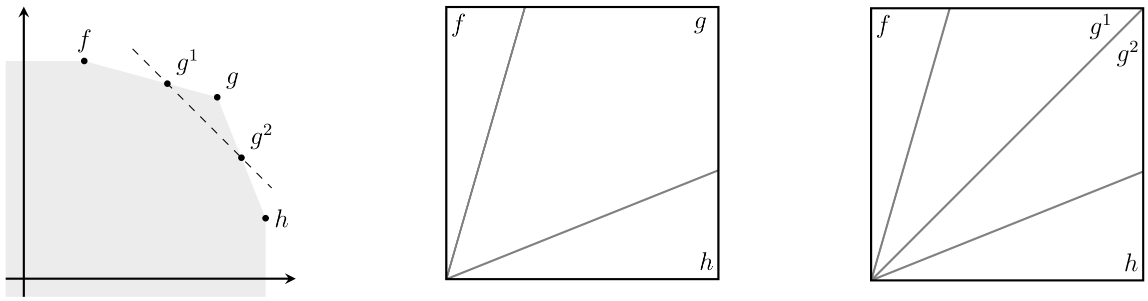
Let $H := \{\lambda' \in \mathbb{R}^\Omega : \lambda \cdot \lambda' = \zeta\}$ denote the (unique) hyperplane with normal λ that intersects the boundary (but not the interior) of $T(A)$. The half-space $H^*(\zeta) := \{\lambda' \in \mathbb{R}^\Omega : \lambda \cdot \lambda' \leq \zeta\}$ below H contains $T(A)$. Shifting H^* toward the origin by a small amount (that is, taking $H^*(\zeta')$ with $\zeta' < \zeta$) and intersecting with $T(A)$ yields a new decision polytope T' where one or more vertices of $T(A)$ are split into multiple vertices. This means that for at least one $f \in A$, the vertex $z^f = U(f) \in T(A)$ is split into vertices $z^{f_1}, \ldots, z^{f^n}$ in T', and the set $S^A(f)$ is divided into convex cones $S(f^i) \subseteq S^A(f)$ where $S(f^i) := \{s \in S : s \cdot z^{f^i} > s \cdot z \ \forall z \neq z^{f^i}\}$.

By construction, T' has a face with normal λ. By letting $\zeta' \to \zeta$, T' converges to $T(A)$ (in the Hausdorff metric). Thus, if the vertex $z^f \in T(A)$ corresponding to some $f \in A$ is split into $z^{f_1}, \ldots, z^{f^n}$ in T', the coordinates z^{f^i} each converge to z^f as $\zeta' \to \zeta$. Therefore, acts f^i such that $U(f^i) = z^{f^i}$ can be chosen such that $f^i \to f$ as $\zeta' \to \zeta$. Moreover, the acts corresponding to new vertices can be chosen so that the resulting menu B is independent.

Thus, there is a ζ' near ζ for which the corresponding menu B satisfies $d(f^*, c^\sigma(B)) < \delta$; that is, $c^\sigma(B)$ is in the interior of the ball of radius δ around f^*. Since B is independent, F^B contains a ball of radius δ' around $c^\sigma(B)$ for some $\delta' > 0$. Thus, $\dim(F^A \cap F^B) = \dim(F)$, so that A and B share a representation. $\square$

Lemma 17. *Suppose A is a k-menu and $B \subseteq A$ such that $c^e(A) \in B$. There exists an experiment σ such that for every $f \in B$, σ contains a signal s^f such that $c^{s^f}(A) = f$.*

Proof. Let $f^e \in B$ denote the act satisfying $c^e(A) \in B$. For each $f \in B \backslash f^e$, pick s^f such that $c^{s^f}(A) = f$; such s^f exist because A is a k-menu. Let $s := \sum_{f \in B \backslash f^e} s^f$, and choose $\alpha \in (0, 1)$ such that $e - \alpha s \in S^A(f^e)$. Such an α exists because for small enough α, $e - \alpha s$ is close to $e \in S^A(f^e)$, which is a full-dimensional subset of S. Then $\sigma := [\alpha s^f : f \in B \backslash f^e] \cup [e - \alpha s]$ is a well-defined experiment satisfying the desired properties. $\square$



(a) Utility Space

(b) Signal space for $A = \{f, g, h\}$

(c) Signal space for $B = \{f, g^1, g^2, h\}$

Figure 9: Illustration of Lemma 16. The shaded region in (a) is $T(A)$. T' is formed by clipping off the region above the dashed line, effectively replacing coordinate g with coordinates g^1 and g^2. The region in signal space where g is chosen from $A = \{f, g, h\}$ is divided into regions for g^1 and g^2 in menu $B = \{f, g^1, g^2, h\}$ (in this example, the dashed line is orthogonal to e). If the acts yielding utility coordinates g^1 and g^2 are sufficiently close to g, then A-interior experiments σ yield induced acts $c^\sigma(A)$ and $c^\sigma(B)$ that are close to each other.

Lemma 18. *Suppose U is a k-utility profile and U' is an ℓ-utility profile such that $T = T(U)$ and $T' = T(U')$ satisfy $\frac{1}{W}e \in \mathcal{N}(T) \cap \mathcal{N}(T')$. For each choice of A and B such that $U = U(A)$ and $U' = U(B)$, there exists an N-utility profile U^* and $\lambda \in \mathbb{R}^\Omega$ such that:*

(i) $U \cup U^*$ is a $(k+N)$-utility profile and $U' \cup (U^* + \lambda)$ is a $(\ell + N)$-utility profile,

(ii) There exists $z \in U^*$ such that $e \in S^{U \cup U^*}(z)$ and $e \in S^{U' \cup (U^* + \lambda)}(z + \lambda)$, and

(iii) If $U^* = U(A^*)$ and $U^* + \lambda = U(B^*)$, then A inherits a representation from $A \cup A^*$ and B inherits a representation from $B \cup B^*$.

Proof. Let A and B satisfy $U = U(A)$ and $U' = U(B)$. Choose an A-interior experiment σ and a corresponding neighborhood B^ε, and a B-interior σ' with neighborhood $B^{\varepsilon'}$. As in the proof of Lemma 12, the half-spaces corresponding to signals $s \in \hat{\sigma} \in B^\varepsilon$ passing through the point $U(f^s)$ (where $f^s = c^s(A)$) intersect to form a space $T^*(A)$ such that $T(A) \subseteq T^*(A)$. Moreover, $T^*(A) \backslash T(A)$ contains a full-dimensional subset of $\mathbb{R}^\Omega$ near the face of $T(A)$ with normal e because every $s \in \hat{\sigma} \in B^\varepsilon$ is bounded away from e. A similar argument yields a region $T^*(B)$ for which analogous statements hold. Thus, there is a $\delta > 0$ such that both $T^*(A) \backslash T(A)$ and $T^*(B) \backslash T(B)$ contain an open ball of radius δ. Letting D^A denote such a ball in $T^*(A) \backslash T(A)$ and D^B the ball in $T^*(B) \backslash T(B)$, we get $D^B = D^A + \lambda$ for some $\lambda \in \mathbb{R}^\Omega$.

We now construct U^*. First, pick a point $z^1 \in D^A$. Then $z^1 + \lambda \in D^B$. By our choice of D^A and D^B, we have that $T(U \cup \{z^1\})$ is a $(k+1)$-polytope where $e \in S^{U \cup \{z^1\}}(z^1)$; that is, if some act f^1 satisfies $U(f^1) = z^1$, then $c^e(A \cup \{f^1\}) = f^1$. Since this is a strict preference, there is a full-dimensional, convex set of signals s such that $c^s(A \cup \{f^1\}) = f^1$, and e belongs to the interior of this set. Similar statements hold for $B \cup \{g^1\}$ for any g^1 such that $U(g^1) = z + \lambda$. Therefore, there is a full-dimensional set of signals s such that $c^s(A \cup \{f^1\}) = f^1$ and $c^s(B \cup \{g^1\}) = g^1$. Call the set of all such s the *support* of z^1.

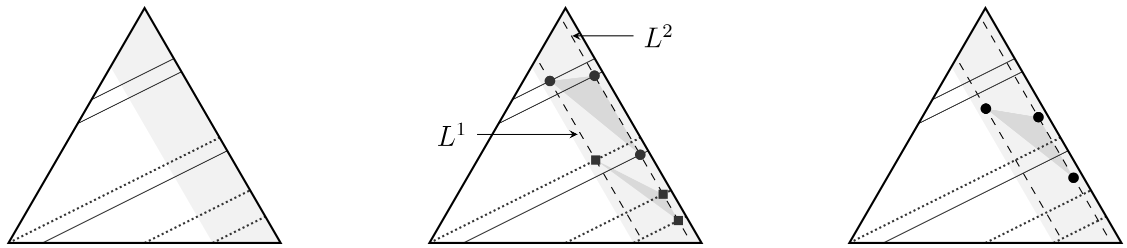
We now proceed by induction. Suppose $U^* = \{z^1, \ldots, z^n\} \subseteq D^A$ such that each $z \in U^*$ has full-dimensional support. That is, for any A^* such that $U(A^*) = U^*$ and $f \in A^*$, the set $S^z = S^{A \cup A^*}(f) \cap S^{B \cup (A^* + \lambda)}(g)$ has full dimension, where $g \in B^*$ satisfies $U(g) = U(f) + \lambda$. Pick any $z \in U^*$ and s in the interior of S^z such that $s \neq \lambda e$ for all λ. Let $H(s; z)$ denote the hyperplane with normal s passing through z. If $z^{n+1} \in H(s; z) \backslash z$ is sufficiently close to z, then $z^{n+1} \in D^A$, $T(U \cup U^* \cup \{z^{n+1}\})$ is a $(k+n+1)$-polytope, and $T(U' \cup (U^* \cup \{z^{n+1}\} + \lambda))$ is an $(\ell + n + 1)$-polytope. Moreover, z^{n+1} has full-dimensional support.

The resulting set $U^* = \{z^1, \ldots, z^N\}$ clearly satisfies (i) and (ii). For (iii), note that our original choice of D^A and D^B guarantees that for all $s \in \hat{\sigma} \in B^{\varepsilon}$, $c^s(A \cup A^*) = c^s(A)$ and $s' \in \hat{\sigma}' \in B^{\varepsilon'}$ implies $c^{s'}(B \cup B^*) = c^{s'}(B)$. Thus, $F^A(B^{\varepsilon}) \subseteq F^{A \cup A^*}$ and $F^B(B^{\varepsilon}) \subseteq F^{B \cup B^*}$, so that $\dim(F^A) \leq \dim(F^{A \cup A^*})$ and $\dim(F^B) \leq \dim(F^{B \cup B^*})$. $\square$

Lemma 19. *Suppose $U, U' \subseteq (0,1)$ are sets of cardinality N. There exist $P, Q \subseteq \Delta X$ and a bijection $\varphi : P \to Q$ such that φ is oriented, $U = \{u(p) : p \in P\}$, and $U' = \{u(q) : q \in Q\}$.*

Proof. Figure 10 illustrates the idea of the proof. Consider the indifference curves (hyperplanes) in ΔX corresponding to the utilities in $U \cup U'$. There is an edge E of ΔX such that each of these planes intersects the (relative) interior of E. Specifically, E is any edge connecting lotteries δ_b and δ_w for any choice of $b, w \in X$ such that $u(b) \geq u(x) \geq u(w)$ for all $x \in X$. Since each utility level is interior, it can be expressed as a non-degenerate mixture of $u(b)$ and $u(w)$, forcing the associated hyperplane to intersect the relative interior of E. Parallel to this edge is an interior line L passing through (the interior of) each hyperplane, so that in fact there is an $\varepsilon > 0$ such that every parallel ε perturbation of L passes through each hyperplane. Let $B \subseteq \Delta X$ denote the region spanned by these perturbations; clearly, B has dimension equal to that of ΔX (namely, $N-1$).

Pick $N-1$ lines $L^1, \ldots, L^{N-1}$ in B, each parallel to L, such that the convex hull of $\{L^1, \ldots, L^{N-1}\}$ has dimension $N-1$. Order the $u^i \in U$ so that $u^1 > u^2 > \ldots > u^N$. For $i = 1, \ldots, N-1$, let p^i be the (unique) intersection of L^i and the indifference plane for utility u^i, and let p^N be the unique intersection of L^{N-1} with the indifference plane for utility u^N. Observe that $\{p^1, \ldots, p^{N-1}\}$ lie on a hyperplane H in ΔX and that p^N is not in the affine hull of H because L^{N-1} passes through H at a single point (p^{N-1}) while p^N lies at a different point on L^{N-1}. Thus, $P = \{p^1, \ldots, p^N\}$ is affinely independent. The same lines $L^1, \ldots, L^{N-1}$ and order-based construction for U' yield an affinely independent set $Q = \{q^1, \ldots, q^N\}$ where $u(q^1) > \ldots > u(q^N)$.



(a) Utility Levels (b) P (dots) and Q (squares) (c) P^α, $\alpha = 0.5$

Figure 10: Illustration of Lemma 19. Solid lines represent utility levels for U, and dotted lines for U'. The shaded region in (a) is the region $B \subseteq \Delta X$ referenced in the proof. With this construction, every set P^α is affinely independent.

Now consider $P^\alpha := \{(1-\alpha)p^i + \alpha q^i : i = 1, \dots, N\}$. Observe that $(1-\alpha)u(p^i) + \alpha u(q^i) > (1-\alpha)u(p^{i+1}) + \alpha u(q^{i+1})$ for all $i = 1, \dots, N-1$ because $u(p^i) > u(p^{i+1})$ and $u(q^i) > u(q^{i+1})$. Notice also that $(1-\alpha)p^i + \alpha q^i$ is on line L^i ($i = 1, \dots, N-1$) and $(1-\alpha)p^N + \alpha q^N$ is on L^{N-1}. Thus, by the same argument, P^α is affinely independent. Hence, the map $\varphi : P \to Q$ given by $\varphi(p^i) = q^i$ ($i = 1, \dots, N$) is oriented. □

Lemma 20. *If A and B are independent, then A and B share a representation.*

Proof. By Lemma 16, we may assume that $e \in \mathcal{N}(A)$ and $e \in \mathcal{N}(B)$. Then, by Lemma 18, there is a utility profile U and a $\lambda \in \mathbb{R}^\Omega$ such that if $U = U(A^*)$ and $U' := U + \lambda = U(B^*)$, then A and $A' := A \cup A^*$ share a representation, and B and $B' := B \cup B^*$ share a representation. In fact, by Lemma 17, A' shares a representation with A^* provided A^* is independent. Similarly, B' shares a representation with B^* provided B^* is independent. Therefore, it will suffice to find independent menus A^* and B^* such that $U = U(A^*)$, $U' = U(B^*)$, and such that A^* and B^* share a representation.

To do so, choose a state ω and apply Lemma 19 to the sets $U_\omega := \{z_\omega : z \in U\}$ and $U'_\omega := \{z'_\omega : z' \in U'\}$ to get affinely independent sets $P_\omega := \{p^z_\omega : z \in U\}$ and $Q_\omega := \{q^{z'}_\omega : z' \in U'\}$ such that $u(p^z_\omega) = z_\omega$ and $u(q^{z'}_\omega) = z'_\omega$ for all $z \in U$ and $z' \in U'$ (if necessary, apply a small perturbation to U and U' in order to get N distinct utility values in U_ω for each ω, and N distinct utility values in U'_ω for all ω). Repeating this for each ω yields acts $f^z := (p^z_\omega)_{\omega\in\Omega}$ and $g^{z'} := (q^{z'}_\omega)_{\omega\in\Omega}$ for each $z \in U$ and $z' \in U'$. Then $A^* := \{f^z : z \in U\}$ and $B^* := \{g^{z'} : z' \in U'\}$ are oriented, so that by Lemma 15, A^* and B^* share a representation. □

Lemma 21. *There is a unique, linear $L^* : F \to \mathbb{R}$ such that, for all k-menus A, the function $\sigma \mapsto L^*(c^\sigma(A))$ represents $\succsim^A$ on $\sigma \in \mathcal{E}^c(A)$.*

Proof. By Lemma 20, all independent menus share a representation. This means there is a unique linear $\succsim^*$ on F that agrees with each relation $\succsim^B$ where B is independent. This

$\succsim^*$ also agrees with $\succsim^A$ since every k-menu inherits a representation from an independent menu (Lemma 13). To construct L^*, choose any independent menu A and consider the linear representation $W^A : F^A \to \mathbb{R}$ constructed at the start of Step 1. Since F^A has full dimension, W^A has a unique linear extension to F. Take L^* to be this extension. $\square$

Proof of Theorem 1

Theorem 1A. *Suppose c has a Bayesian Representation. Let $\dot{\succsim} = (\dot{\succsim}^A)_{A\in\mathcal{A}}$ where $\dot{\succsim}^A$ is the restriction of $\succsim^A$ to $\mathcal{E}^c(A)$. Then $(\dot{\succsim}, c)$ satisfies Axioms 1.1–1.6 if and only if there exists a full-support $\nu \in \Delta\Omega$ and a non-constant utility index $v : X \to \mathbb{R}$ such that, for all $A \in \mathcal{A}$ and all $\sigma, \sigma' \in \mathcal{E}^c(A)$,*

$$\sigma \succsim^A \sigma' \Leftrightarrow \sum_{\omega\in\Omega} \nu_\omega \sum_{s\in\sigma} s_\omega v(c^s_\omega(A)) \geq \sum_{\omega\in\Omega} \nu_\omega \sum_{s'\in\sigma'} s'_\omega v(c^{s'}_\omega(A)).$$

Moreover, ν is unique and v is unique up to positive affine transformation.

Proof. Let L^* be the linear representation given by Lemma 21 and let $\sigma, \sigma' \in \mathcal{E}^c(A)$. Then there is a submenu $A' \subseteq A$ that is a k-menu (for some k) such that $c^\sigma(A) = c^\sigma(A')$ and $c^{\sigma'}(A) = c^{\sigma'}(A')$. By the Consistency axiom, $\sigma \succsim^A \sigma'$ if and only if $\sigma \succsim^{A'} \sigma'$. Thus, $\sigma \succsim^A \sigma'$ if and only if $L^*(c^\sigma(A)) \geq L^*(c^{\sigma'}(A))$.

By the Non-Degeneracy axiom, L^* must be non-constant; otherwise, by the previous paragraph, every $\succsim^A$ assigns indifference among all experiments in $\mathcal{E}^c(A)$. Thus, by Lemma 8, $\succsim^{A^*}$ (uniquely) extends to $\succsim^*$ on F (where A^* is the symmetric menu constructed in Step 1), and $\succsim^*$ satisfies all of the Anscombe-Aumann axioms, including Non-Degeneracy. Thus, $\succsim^*$ has an expected utility representation with unique ν and unique (up to positive affine transformation) utility index v. Since L^* is a linear representation for $\succsim^*$, it follows that the expected utility representation holds for all menus $\succsim^A$ on $\mathcal{E}^c(A)$. $\square$

We now prove Theorem 1. First, we verify that existence of the representation implies Axiom 1.7. Let $V(f) := \sum_{\omega\in\Omega} v(f_\omega)\nu_\omega$. If $A^n \to^c A$, then $V(c^\sigma(A^\infty))$ is well-defined and denotes the expected utility of the "induced act" under the selection implied by $A^n \to^c A$.

If $V^A(\sigma) \geq V^A(\sigma')$, then the best-case selection for σ is weakly better than the best-case selection for σ'. So, it will suffice to show that there is a sequence $A^n \to^c A$ such that $V(c^\sigma(A^\infty)) = V^A(\sigma)$ for all σ; in other words, that there is a sequence $A^n \to^c A$ giving rise to the Sender-optimal selection. Then, employing $\hat{A} = A^*$ from Step 1 of the proof of Theorem 1A, one can find the desired $\hat{\sigma}, \hat{\sigma}'$, α, and h.

We construct the sequence $(A^n)_{n=1}^\infty$ by specifying utility coordinates for the acts, then selecting acts yielding those coordinates. Enumerate the acts in A as $f^1, \ldots, f^K$. Choose sequences $\varepsilon_n^k \to 0$ such that $\varepsilon_n^k > 0$ for all n and $k > \ell \Rightarrow \varepsilon_n^k > \varepsilon_n^\ell$ for all n. For each $f^k \in A$, let $\tilde{u}^k := \left(\frac{\nu_\omega}{\mu_\omega} v(f_\omega^k)\right)_{\omega\in\Omega}$. Choose a sequence $\lambda_n \to 0$ such that $\lambda_1 = 1$ and $0 < \lambda_n < 1$ for all n. For each n, let $\tilde{u}^{k,n} := (1 - \lambda_n)u(f^k) + \lambda_n[\tilde{u}^k + \varepsilon_n^k]$, where $u(f^k) := (u(f_\omega^k))_{\omega\in\Omega}$. Then $\tilde{u}^{k,n} \to u(f^k)$ as $n \to \infty$. Finally, choose a sequence $\alpha_n \to 1$ such that $0 < \alpha_n < 1$ and $u^{k,n} := \alpha_n \tilde{u}^{k,n} \in [0,1]^\Omega$ for all k, n; such a sequence exists because $\tilde{u}^{k,n} \to u(f^k)$ and the range of u is $[0,1]$. Since $u^{k,n} \in [0,1]^\Omega$ and $u^{k,n} \to u(f^k)$ as $n \to \infty$, there exist acts $f^{k,n}$ such that $u(f^{k,n}) = u^{k,n}$ and $f^{k,n} \to f^k$. Let $A^n := \{f^{k,n} : f^k \in A\}$. It is straightforward to verify that (i) if $c^s(A) = f^k$, then $c^s(A^n) \to f^k$, and (ii) if $c^s(A)$ is multi-valued, then $c^s(A^n)$ converges to the f^k (with largest k) among those acts in $c^s(A)$ that maximize expected utility under (ν^s, v). Thus, $A^n \to^c A$ and $V(c^\sigma(A^\infty)) = V^A(\sigma)$ for all σ.

To see that Axioms 1.1–1.7 imply the desired representation, once again invoke Theorem 1A to establish the representation on $\mathcal{E}^c(A)$ for all A. Letting $A^n \to^c A$ denote the sequence constructed above, we have $V(c^\sigma(A^\infty)) \geq V(c^\sigma(B^\infty))$ for all $B^n \to^c A$. Thus, applying Axiom 1.7 with $B^n = A^n$ implies that $\sigma \succsim^A \sigma' \Leftrightarrow V^A(\sigma) \geq V^A(\sigma')$.

B Proof of Theorem 2

I prove that if c satisfies Axioms 2.1—2.2, then c has a Bayesian representation (μ, u) (the converse is straightforward). By parts (i) and (ii) of Axiom 2.1, each c^s has a (unique) complete, transitive, and non-degenerate rationalizing preference relation $\succsim^s$.

Lemma 22. *If $f \succ^s g$ and $\alpha \in (0,1)$, then $\alpha f + (1-\alpha)h \succ^s \alpha g + (1-\alpha)h$ for all $h \in F$.*

Proof. Let $A = \{f, g\}$ and $B = \{h\}$. Since $f \succ^s g$, we have $c^s(A) = \{f\}$. Thus, $\alpha c^s(A) + (1-\alpha)c^s(B) = \{\alpha f + (1-\alpha)h\}$, so that (by part (iii) of Axiom 2.1) $c^s(\alpha A + (1-\alpha)B) = \{\alpha f + (1-\alpha)h\}$. Since $\alpha g + (1-\alpha)h \in \alpha A + (1-\alpha)B$, it follows that $\alpha f + (1-\alpha)h \succ^s \alpha g + (1-\alpha)h$. □

Lemma 23. *Each $\succsim^s$ is continuous (that is, weak contour sets are closed).*

Proof. Each c^s is closed-valued and, by Axiom 2.1(iv), upper hemicontinuous. Thus, c^s has the closed-graph property: if $A^n \to A$, $f^n \to f$, and $f^n \in c^s(A^n)$ for all n, then $f \in c^s(A)$.

To see that upper contour sets of $\succsim^s$ are closed, fix g and suppose $f^n \to f$ where $f^n \succsim^s g$ for all n. Then $f^n \in c^s(\{g, f^n\})$ for all n. Clearly, $\{g, f^n\} \to \{g, f\}$. Thus, by the closed-graph property, $f \in c^s(\{g, f\})$, so that $f \succsim^s g$.

For the lower contour sets, fix g and suppose $f^n \to f$ where $g \succsim^s f^n$ for all n. Letting $g^n = g$ for all n, it follows that $g^n \in c^s(\{g^n, f^n\})$ for all n. Clearly, $g^n \to g$ and $\{g^n, f^n\} \to \{g, f\}$. Thus, by the closed-graph property, $g \in c^s(\{g, f\})$, so that $g \succsim^s f$. □

Lemma 24. *If $p[\omega]h \succsim^s q[\omega]h$ and $s_\omega, s'_{\omega'} > 0$, then $p[\omega']h' \succsim^{s'} q[\omega']h'$ for all $h' \in F$.*

Proof. Let $L = \{p, q\}$. Since $p[\omega]h \succsim^s q[\omega]h$, we have $p[\omega]h \in c^s(L[\omega]h)$. Thus, by part (v) of Axiom 2.1, we have $p[\omega']h' \in c^{s'}(L[\omega']h')$, so that $p[\omega']h' \succsim^{s'} q[\omega']h'$. $\square$

By Lemmas 22–24, each $\succsim^s$ satisfies the Anscombe-Aumann axioms and, hence, can be represented by expected utility with prior μ^s and (non-constant) utility index u^s. The state independence axiom expressed by Lemma 24 implies μ^e (where $e = (1, \ldots, 1) \in S$) has full support, and that for all $s, s' \in S$, u^s is a positive affine transformation of $u^{s'}$. Thus, we may assume $u^s = u := u^e$ for all s. To complete the proof, we verify that μ^s is the Bayesian posterior of $\mu := \mu^e$ conditional on s; that is, that $f \succsim^s g \Leftrightarrow \sum_\omega u(f_\omega)s_\omega\mu^e_\omega \geq \sum_\omega u(g_\omega)s_\omega\mu^e_\omega$. Notice that $f \succsim^s g$ if and only if $ef + (1-e)h \succsim^s eg + (1-e)h$, which (by Axiom 2.2) holds if and only if $sf + (1-s)h \succsim^e sg + (1-s)h$. By the expected utility representation for $\succsim^e$, $sf + (1-s)h \succsim^e sg + (1-s)h \Leftrightarrow \sum_\omega u(f_\omega)s_\omega\mu^e_\omega \geq \sum_\omega u(g_\omega)s_\omega\mu^e_\omega$, as desired.

C Proofs for Sections 4 and 5

Lemma 25. *Suppose $\succsim$ is representable by (ν, μ, v, u). Let A be a pq-bet.*

(i) If $v(p) > v(q)$ and $u(p) \geq u(q)$, then σ^* is top-ranked by $\succsim^A$; if instead $u(q) > u(p)$, then σ^* is bottom-ranked by $\succsim^A$.

(ii) $\succsim^A$ is degenerate if and only if $v(p) = v(q)$.

(iii) v and u agree on the ranking of p and q if and only if σ^* is top-ranked by $\succsim^A$.

Proof. Note that for every σ, there exists $\alpha^\sigma \in [0,1]$ such that $V^A(\sigma) = \alpha^\sigma v(p) + (1-\alpha^\sigma)v(q)$. To prove (i), suppose $v(p) > v(q)$. If $u(p) \geq u(q)$, then $\alpha^{\sigma^*} = 1$ and so $\sigma^* \succsim^A \sigma$ for all σ. If $u(q) > u(p)$, then $\alpha^{\sigma^*} = 0$ and $\sigma \succsim^A \sigma^*$ for all σ. For (ii), observe that if $v(p) \neq v(q)$, then $\sigma^* \not\sim^A e$ because $\alpha^e \in (0,1)$ while $\alpha^{\sigma^*} \in \{0,1\}$. For (iii), suppose v and u do not agree on the ranking of p and q: without loss of generality, $v(p) > v(q)$ and $u(q) > u(p)$. Then σ^* is bottom-ranked by (i) and $\succsim^A$ is non-degenerate by (ii), so σ^* is not top-ranked. $\square$

C.1 Proof of Theorem 3

Lemma 26. *Either $(v, u) \approx (\dot{v}, \dot{u})$ or $(v, u) \approx (-\dot{v}, -\dot{u})$.*

Proof. Let $p \in \Delta X$. By Lemma 25, v and u agree on the ranking of p and q if and only if σ^* is top-ranked by $\succsim^A$ for all pq-bets A. Thus, $\{q : v$ and u agree on the ranking of p and $q\} = \{q : \sigma^*$ is top-ranked by $\succsim^A \; \forall pq$-bets $A\} = \{q : \dot{v}$ and $\dot{u}$ agree on the ranking of p and $q\}$. This set is determined by two (possibly identical) planes through p corresponding to indifference curves for the utility indices. By Lemma 25, the indifference curve for v (and $\dot{v}$) is the set $\{q : \succsim^A \text{ is degenerate } \forall pq\text{-bets } A\}$, so the indifference curve for u (and $\dot{u}$) is the other plane. Thus, $v \approx \dot{v}$ or $v \approx -\dot{v}$ because v and $\dot{v}$ have the same indifference curves. Similarly, $u \approx \dot{u}$ or $u \approx -\dot{u}$. However, only two combinations can be consistent with the agreement region: either $(v, u) \approx (\dot{v}, \dot{u})$ or $(v, u) \approx (-\dot{v}, -\dot{u})$. □

Lemma 27. $\mu = \dot{\mu}$.

Proof. If $s, t \in \sigma$, let σ^{s+t} denote an experiment formed by replacing s and t with a single column $s+t$. Signals s, t are *EF-equivalent* if there are neighborhoods $N(s), N(t)$ such that $\sigma \sim^A \sigma^{s'+t'}$ for all EF-bets A and all σ such that $s', t' \in \sigma$, $s' \in N(s)$, and $t' \in N(t)$.

First, we prove that s, t are EF-equivalent if and only if either $[\mu^s(E) > \mu^s(F)$ and $\mu^t(E) > \mu^t(F)]$ or $[\mu^s(F) > \mu^s(E)$ and $\mu^t(F) > \mu^t(E)]$ (that is, *μ^s and μ^t agree on the ranking of E and F*). It is straightforward to show that if μ^s and μ^t agree on the ranking of E and F, then s and t are EF-equivalent. Conversely, suppose s and t are EF-equivalent. Suppose toward a contradiction that μ^s and μ^t do not agree on the ranking of E and F. Then, for all $\varepsilon > 0$, there exist $s' \in N^\varepsilon(s), t' \in N^\varepsilon(t)$ such that (without loss of generality) $\mu^{s'}(E) > \mu^{s'}(F)$, $\mu^{t'}(F) > \mu^{t'}(E)$, $\mu^{s'+t'}(E) \neq \mu^{s'+t'}(F)$, $\nu^{s'}(E) \neq \nu^{s'}(F)$, and $\nu^{t'}(E) \neq \nu^{t'}(F)$. Let $A = \{pEq, pFq\}$ be a bet such that $u(p) \neq u(q)$ and $v(p) \neq v(q)$ (thus, $\dot{u}(p) \neq \dot{u}(q)$ and $\dot{v}(p) \neq \dot{v}(q)$ by Lemma 26). Without loss of generality, suppose $u(p) > u(q)$.

Let $\sigma = [r', s', t']$, where $r' = e - s' - t'$, so that $\sigma^{s'+t'} = [r', s' + t']$. We may assume $r'_\omega > 0$ for all ω (if necessary, scale s' and t' down by a factor $\lambda > 0$), so $r' \in S$. By definition, there are acts $f^{\hat{s}} \in c^{\hat{s}}(A)$ such that, for all $\hat{\sigma}$, $V^A(\hat{\sigma}) = \sum_{\hat{s} \in \hat{\sigma}} V^A(\hat{s})$, where $V^A(\hat{s}) := \sum_{\omega \in \Omega} \hat{s}_\omega \nu_\omega v(f^{\hat{s}}_\omega)$. Thus, $V^A(\sigma) = V^A(r') + V^A(s') + V^A(t')$ and $V^A(\sigma^{s'+t'}) = V^A(r') + V^A(s' + t')$. So, $\sigma \sim^A \sigma^{s'+t'}$ if and only if $V^A(s') + V^A(t') = V^A(s' + t')$. We have

$$V^A(s') + V^A(t') = v(p) \left[\sum_{\omega \in E} s'_\omega \nu_\omega + \sum_{\omega \in F} t'_\omega \nu_\omega \right] + v(q) \left[\sum_{\omega' \in E^c} s'_{\omega'} \nu_{\omega'} + \sum_{\omega' \in F^c} t'_{\omega'} \nu_{\omega'} \right]$$

because $u(p) > u(q)$, $\mu^{s'}(E) > \mu^{s'}(F)$, and $\mu^{t'}(F) > \mu^{t'}(E)$. There are two cases:

1. $\mu^{s'+t'}(E) > \mu^{s'+t'}(F)$. Then $c^{s'+t'}(A) = pEq$, so

$$V^A(s' + t') = \sum_{\omega \in E} (s'_\omega + t'_\omega) \nu_\omega v(p) + \sum_{\omega' \in E^c} (s'_{\omega'} + t'_{\omega'}) \nu_{\omega'} v(q)$$

and $\sigma \sim^A \sigma^{s'+t'}$ if and only if

$$v(p) \sum_{\omega \in F} t'_\omega \nu_\omega + v(q) \sum_{\omega' \in F^c} t'_{\omega'} \nu_{\omega'} = v(p) \sum_{\omega \in E} t'_\omega \nu_\omega + v(q) \sum_{\omega' \in E^c} t'_{\omega'} \nu_{\omega'}.$$

Since $v(p) \neq v(q)$, this is equivalent to $\nu^{t'}(E) = \nu^{t'}(F)$, a contradiction.

2. $\mu^{s'+t'}(F) > \mu^{s'+t'}(E)$. Similar logic yields $\sigma \not\sim^A \sigma^{s'+t'}$ since $\nu^{s'}(E) \neq \nu^{s'}(F)$.

Thus, in each case, we have $\sigma \not\sim^A \sigma^{s'+t'}$, contradicting EF-equivalence.

We have shown that μ^s and μ^t agree on the ranking of E and F if and only if s, t are EF-equivalent. Thus, the same holds for $\dot{\mu}^s$ and $\dot{\mu}^t$. Therefore $\{s : \mu^s(E) = \mu^s(F)\} = \{s : \dot{\mu}^s(E) = \dot{\mu}^s(F)\}$, which is a hyperplane H^{EF} in S. Let $E = \{\omega\} \neq \{\omega'\} = F$. Then $s \in H^{EF}$ satisfies $s_\omega \mu_\omega = s_{\omega'} \mu_{\omega'}$, pinning down the likelihood ratio $\frac{\mu_\omega}{\mu_{\omega'}} = \frac{s_\omega}{s_{\omega'}}$ (since μ has full support, there is $s \in H^{EF}$ such that $s_\omega \neq 0 \neq s_{\omega'}$). Since probability distributions are uniquely determined by their likelihood ratios, it follows that $\mu = \dot{\mu}$. □

Lemma 28. *$\nu = \dot{\nu}$.*

Proof. Let $A = \{pEq, pFq\}$ be a bet such that $u(p) \neq u(q)$ and $v(p) \neq v(q)$ (so $\dot{u}(p) \neq \dot{u}(q)$ and $\dot{v}(p) \neq \dot{v}(q)$ by Lemma 26). Consider an experiment $\sigma = [s, t]$ where s, t belong to the interior of S, $\mu^s(E) > \mu^s(F)$, and $\mu^t(F) > \mu^t(E)$. Then, for $\|\delta\|$ sufficiently small, the experiment $\sigma^\delta = [s + \delta, t - \delta]$ is well-defined and satisfies $\mu^{s+\delta}(E) > \mu^{s+\delta}(F)$ and $\mu^{t-\delta}(F) > \mu^{t-\delta}(E)$. It follows that $\sigma \sim^A \sigma^\delta$ if and only if

$$\sum_{\omega \in E} \nu_\omega \delta_\omega = \sum_{\omega \in F} \nu_\omega \delta_\omega \tag{4}$$

because $u(p) \neq u(q)$ and $v(p) \neq v(q)$. This also holds for $\dot{\nu}$. Consider the case $F = E^c$. Then every δ satisfying (4) belongs to the kernel of the linear transformation $L : \delta \mapsto \sum_{\omega \in E} \nu_\omega \delta_\omega - \sum_{\omega \in E^c} \nu_\omega \delta_\omega$. The set of all such δ is a hyperplane with normal vector determined by ν. We can elicit all such δ within a neighborhood of 0. This is a full-dimensional subset of the hyperplane and, hence, sufficient to reveal the normal vector. Thus, $\nu = \dot{\nu}$. □

Lemma 29. *$v \approx \dot{v}$ and $u \approx \dot{u}$.*

Proof. Pick $p^1, p^2, p^3 \in \Delta X$ such that $u(p^3) > u(p^2) > u(p^1)$, $u(p^2) - u(p^1) > u(p^3) - u(p^2)$, and $v(p^3) \neq v(p^1)$. Pick any $\omega \neq \omega'$, $q \in \Delta X$, and $h' \in F$, and let $A = \{f, g, h\}$ where

$$\begin{aligned}
f_\omega &= \mu_{\omega'} p^1 + (1 - \mu_{\omega'}) q & f_{\omega'} &= \mu_\omega p^3 + (1 - \mu_\omega) q & f_{\hat{\omega}} &= h'_{\hat{\omega}} \quad \forall \hat{\omega} \neq \omega, \omega' \\
g_\omega &= \mu_{\omega'} p^2 + (1 - \mu_{\omega'}) q & g_{\omega'} &= \mu_\omega p^2 + (1 - \mu_\omega) q & g_{\hat{\omega}} &= h'_{\hat{\omega}} \quad \forall \hat{\omega} \neq \omega, \omega' \\
h_\omega &= \mu_{\omega'} p^3 + (1 - \mu_{\omega'}) q & h_{\omega'} &= \mu_\omega p^1 + (1 - \mu_\omega) q & h_{\hat{\omega}} &= h'_{\hat{\omega}} \quad \forall \hat{\omega} \neq \omega, \omega'.
\end{aligned}$$

Let $c^{\hat{s}}(A)$ denote Receiver's choice from A at $\hat{s}$ under (μ, u). Then $c^s(A) = A$ if $s_\omega = 0 = s_{\omega'}$; $c^s(A) = f$ if $\frac{s_{\omega'}}{s_\omega} > \frac{u(p^2) - u(p^1)}{u(p^3) - u(p^2)}$; $c^s(A) = h$ if $\frac{s_{\omega'}}{s_\omega} < \frac{u(p^3) - u(p^2)}{u(p^2) - u(p^1)}$; and $c^s(A) = g$ if $\frac{u(p^3) - u(p^2)}{u(p^2) - u(p^1)} <$

$\frac{s_{\omega'}}{s_\omega} < \frac{u(p^2)-u(p^1)}{u(p^3)-u(p^2)}$ (these inequalities assume $s_\omega > 0$; at signal $e^{\omega'}$, Receiver chooses f). Thus, there is an interval of values for $\frac{s_{\omega'}}{s_\omega}$ where g is the unique choice. In particular, $c^e(A) = g$.

Let $\sigma = [s,t]$ where $s = \frac{1}{2}e + \delta$ and $t = \frac{1}{2}e - \delta$ for some $\delta \in \mathbb{R}^\Omega$ where $\delta_{\hat\omega} = 0$ for all $\hat\omega \neq \omega, \omega'$. Then $c^s(A) = c^t(A) = g$ (hence, $c^\sigma(A) = g$) for $\|\delta\|$ sufficiently small. Now let $\sigma' = [s', t']$ where $s' = s + \delta'$ and $t' = t - \delta'$. For $\|\delta'\|$ sufficiently small, we once again have $c^{\sigma'}(A) = g$, so that $\sigma \sim^A \sigma'$. We may assume $\nu_{\omega'}\mu_\omega\delta'_{\omega'} \neq \nu_\omega\mu_{\omega'}\delta'_\omega$.

Now consider behavior under $(-v, -u)$. Let $\tilde U^s(\hat f) := -\sum_{\omega\in\Omega} u(\hat f_\omega)\mu_\omega$ denote Receiver's expected utility of $\hat f$ at s under $(\mu, -u)$. Then $\tilde U^s(f) > \tilde U^s(g)$ if and only if $s_\omega(u(p^2)-u(p^1)) > s_{\omega'}(u(p^3)-u(p^2))$ and $\tilde U^s(g) \geq \tilde U^s(h)$ if and only if $s_\omega(u(p^3)-u(p^2)) \geq s_{\omega'}(u(p^2)-u(p^1))$. Since $u(p^2)-u(p^1) > u(p^3)-u(p^2)$, then, $\tilde U^s(g) \geq \tilde U^s(h)$ implies $\tilde U^s(f) > \tilde U^s(g)$. Thus, g is never chosen under $(\mu, -u)$. Similar algebra establishes that $\tilde U^s(f) \geq \tilde U^s(h) \Leftrightarrow s_\omega \geq s_{\omega'}$ and $\tilde U^s(h) \geq \tilde U^s(f)$ iff $s_{\omega'} \geq s_\omega$. With $\sigma = [s,t]$ and $\sigma' = [s', t'] = [s+\delta', t-\delta']$ as defined above, we may therefore assume that $\tilde c^s(A) = \tilde c^{s'}(A) = h$ and $\tilde c^t(A) = \tilde c^{t'}(A) = f$. So, the induced act for σ' under $(\mu, -u)$ is given by

$$\tilde c^{\sigma'}_{\hat\omega}(A) = \begin{cases} \mu_{\omega'}[(s_\omega + \delta'_\omega)p^3 + (t_\omega - \delta'_\omega)p^1] + (1-\mu_{\omega'})q & \text{if } \hat\omega = \omega \\ \mu_\omega[(s_{\omega'} + \delta'_{\omega'})p^1 + (t_{\omega'} - \delta'_{\omega'})p^3] + (1-\mu_\omega)q & \text{if } \hat\omega = \omega' \\ h'_{\hat\omega} & \text{if } \hat\omega \neq \omega, \omega' \end{cases}.$$

(Set $\delta' = 0$ to get $\tilde c^\sigma(A)$). Thus, under $(-v,-u)$, we have $\sigma \sim^A \sigma'$ if and only if $\nu_{\omega'}\mu_\omega\delta'_{\omega'}[v(p^1) - v(p^3)] = \nu_\omega\mu_{\omega'}\delta'_\omega[v(p^1) - v(p^3)]$. Since $v(p^1) \neq v(p^3)$, this reduces to $\nu_{\omega'}\mu_\omega\delta'_{\omega'} = \nu_\omega\mu_{\omega'}\delta'_\omega$. But $\nu_{\omega'}\mu_\omega\delta'_{\omega'} \neq \nu_\omega\mu_{\omega'}\delta'_\omega$, so $\sigma \not\sim^A \sigma'$ under $(-v,-u)$. Since $\sigma \sim^A \sigma'$ under (v,u), this means only one pair—(v,u) or $(-v,-u)$—can be consistent with $\succsim$. □

C.2 Proof of Proposition 1

For (i), suppose first that the preferences of $\dot\succsim$ are more aligned than those of $\succsim$. If A is a pq-bet and $\sigma^* \succsim^A \sigma$ for all σ, then v and u agree on the ranking of p and q by Lemma 25; thus, so do $\dot v$ and $\dot u$. Therefore, $\sigma^* \dot\succsim^A \sigma$ for all σ. Conversely, suppose that for all bets A, σ^* is top-ranked by $\dot\succsim^A$ if it is top-ranked by $\succsim^A$. Suppose v and u agree on the ranking of p and q and let A be a pq-bet. Then σ^* is top-ranked by $\succsim^A$ by Lemma 25 and, hence, top-ranked by $\dot\succsim^A$ as well. Thus, $\dot v$ and $\dot u$ agree on the ranking of p and q.

For (ii), Lemma 25 implies that σ^* is top-ranked by $\succsim^A$ for all bets A if $v \approx u$. Conversely, $v \not\approx u$ implies there are lotteries p, q such that $v(p) > v(q)$ and $u(q) > u(p)$. Let A be a pq-bet. By Lemma 25, σ^* is not top-ranked by $\succsim^A$.

C.3 Proof of Proposition 2

A signal s belongs to the *strict EF-agreement region* if [$\mu^s(E) > \mu^s(F)$ and $\nu^t(E) > \nu^t(F)$] or [$\mu^s(F) > \mu^s(E)$ and $\nu^t(F) > \nu^t(E)$]; otherwise, s is in the *EF-disagreement region.* Signal s belongs to the *strict EF-disagreement region* if [$\mu^s(E) > \mu^s(F)$ and $\nu^t(F) > \nu^t(E)$] or [$\mu^s(F) > \mu^s(E)$ and $\nu^t(E) > \nu^t(F)$]; otherwise, s is in the *EF-agreement region.*

Lemma 30.

(i) If the strict EF-disagreement region is empty, then $\succsim^A$ is Blackwell monotone on $\mathcal{E}$ for all EF-bets A.

(ii) If the strict EF-disagreement region is nonempty, then an EF-informative experiment σ is EF-extreme if and only if every $s \in \sigma$ belongs to the strict EF-agreement region.

Proof. For (i), let $A = \{pEq, pFq\}$ be a bet. There is nothing to prove if $v(p) = v(q)$ or $u(p) = u(q)$, so suppose without loss of generality that $v(p) > v(q)$ and $u(p) \neq u(q)$. For all s, $\nu^s(E) \geq \nu^s(F)$ if and only if $\mu^s(E) \geq \mu^s(F)$. Thus, if $u(p) > u(q)$, then $pEq \in c^s(A) \Leftrightarrow \mu^s(E) \geq \mu^s(F) \Leftrightarrow V^s(pEq) \geq V^s(pFq)$, where $V^s(f) := \sum_{\omega\in\Omega} v(f_\omega)\nu^s_\omega$; that is, Receiver's choices at s maximize V^s on A. If instead $u(q) > u(p)$, Receiver's choice(s) at s maximize $-V^s$ on A. In each case, V^A is Blackwell monotone on $\mathcal{E}$ by Blackwell's theorem.

For (ii), suppose first that every $s \in \sigma$ belongs to the strict EF-agreement region. There exists $\varepsilon > 0$ such that if $s' \in \sigma' \in N^\varepsilon(\sigma)$, then s' belongs to the strict EF-agreement region. Let $A = \{pEq, pFq\}$ be non-degenerate; without loss of generality, suppose $v(p) > v(q)$. If $u(p) \geq u(q)$, then Receiver's choices at s' intersect $\text{argmax}_{f\in A} V^{s'}(f)$, so $\succsim^A$ satisfies the Blackwell ordering on $N^\varepsilon(\sigma)$. If $u(q) > u(p)$, then Receiver's choices maximize $-V^{s'}$, so that $\succsim^A$ reverses the Blackwell ordering on $N^\varepsilon(\sigma)$.

For the converse, suppose σ contains a signal in the EF-disagreement region. Let $A = \{pEq, pFq\}$ be an EF-bet such that $v(p) \neq v(q)$ and $u(p) \neq u(q)$. Without loss of generality, suppose $v(p) > v(q)$. Consider the case $u(p) > u(q)$, so that σ^* is top-ranked by $\succsim^A$ (the case $u(q) > u(p)$ is similar). Let $N^\varepsilon(\sigma)$ be a neighborhood of σ. It will suffice to show that there exists $\sigma', \sigma'' \in N^\varepsilon(\sigma)$ such that $\sigma' \sqsupseteq \sigma''$ and $\sigma'' \succ^A \sigma'$.

If every $s \in \sigma$ belongs to the strict EF-disagreement region, then (since σ is EF-informative) there exist $s, t \in \sigma$ such that $\mu^s(E) > \mu^s(F)$ and $\nu^t(F) > \nu^t(E)$. It is then straightforward to construct a garbling σ'' of $\sigma' = \sigma$ such that $\sigma'' \in N(\sigma)$ and $\sigma'' \succ^A \sigma'$.

Now suppose some signal in σ does not belong to the strict EF-disagreement region. First, we show that there exists $\sigma' \in N^\varepsilon(\sigma)$ and $s^*, t^* \in \sigma'$ such that s^* is in the strict EF-disagreement region, t^* is in the strict EF-agreement region, and (without loss of generality) $\mu^{s^*}(E) > \mu^{s^*}(F)$ and $\mu^{t^*}(F) > \nu^{t^*}(E)$. There are two cases.

1. There exists $s \in \sigma$ such that $\mu^s(E) = \mu^s(F)$. Form σ' by replacing s with s^1, s^2 (each near $\frac{1}{2}s$, hence in $N^\varepsilon(s)$) such that $s^1 + s^2 = s$, s^1 belongs to the strict EF-disagreement region, and s^2 belongs to the strict EF-agreement region (since $\mu^s(E) = \mu^s(F)$, this can be done regardless of whether $\nu^s(E) = \nu^s(F)$). Take $s^* = s^1$ and $t^* = s^2$.

2. No $s \in \sigma$ satisfies $\mu^s(E) = \mu^s(F)$. Let $s \in \sigma$ be in the EF-disagreement region. Without loss of generality, $\mu^s(E) > \mu^s(F)$ and $\nu^s(F) \geq \nu^s(E)$. Since σ is EF-informative, the hypothesis of this case implies there exists $t \in \sigma$ such that $\mu^t(F) > \mu^t(E)$. Since not every signal of σ belongs to the strict EF-disagreement region, we may assume at least one of s or t does not belong to the strict EF-disagreement region. There are two (sub)cases.

   a. If one signal (say s) is in the strict EF-disagreement region, then the other (t) is not. Thus, $\nu^t(F) \geq \nu^t(E)$, and σ' is formed by replacing t with t^1, t^2 (each near $\frac{1}{2}t$) such that $t^1 + t^2 = t$ and $\nu^{t^1}(F) > \nu^{t^1}(E)$. Take $s^* = s$ and $t^* = t^1$.

   b. If neither signal is in the strict EF-disagreement region, then $\nu^s(E) = \nu^s(F)$. First, form σ' by replacing s with s^1, s^2 near $\frac{1}{2}s$ such that $s = s^1 + s^2$ and $\nu^{s^1}(F) > \nu^{s^1}(E)$. If t is in the strict EF-agreement region, take $s^* = s^1$ and $t^* = t$. Otherwise, $\nu^t(E) = \nu^t(F)$; modify σ' by replacing t with t^1, t^2 near $\frac{1}{2}t$ such that $t = t^1 + t^2$ and t^1 is in the strict EF-agreement region. Then take $s^* = s^1$ and $t^* = t^1$.

We now construct a garbling $\sigma' \sqsupseteq \sigma'' \in N^\varepsilon(\sigma)$ such that $\sigma'' \succ^A \sigma'$. We have $c^{s^*}(A) = pEq$ and $c^{t^*}(A) = pFq$. Moreover, $\nu^{s^*}(F) > \nu^{s^*}(E)$ because s^* is in the strict EF-disagreement region. We may write $\sigma' = [r^1, \ldots, r^K, t^*, s^*]$. Consider the garbling matrix M given by

$$M = \begin{bmatrix} I_K & 0 & \\ 0 & 1 & 0 \\ & 1-\alpha & \alpha \end{bmatrix}$$

where I_K denotes the $K \times K$ identity matrix. Then $\sigma'' := \sigma M = [r_1, \ldots, r^K, t', s']$ where $t' = t^* + (1-\alpha)s^*$ and $s' = \alpha s^*$. Clearly, $c^{s'}(A) = c^{s^*}(A)$ and, for large enough $\alpha \in (0,1)$, $c^{t'}(A) = c^{t^*}(A)$. Thus, for $\alpha \in (0,1)$ sufficiently large, we have $\sigma'' \in N^\varepsilon(\sigma)$ and

$$\frac{V^A(\sigma'') - V^A(\sigma')}{1-\alpha} = v(p)\left[\sum_{\omega \in F} \nu_\omega s^*_\omega - \sum_{\omega \in E} \nu_\omega s^*_\omega\right] - v(q)\left[\sum_{\omega \in E^c} \nu_\omega s^*_\omega - \sum_{\omega \in F^c} \nu_\omega s^*_\omega\right].$$

Thus, $V^A(\sigma'') - V^A(\sigma') > 0$ if and only if $[v(p) - v(q)][\nu^{s^*}(F) - \nu^{s^*}(E)] > 0$; since $v(p) > v(q)$ and $\nu^{s^*}(F) > \nu^{s^*}(E)$, the proof is complete. □

To prove part (i) of Proposition 2, observe that by Lemma 30(ii), the EF-agreement is pinned down by the set of EF-extreme experiments. Thus, $(\dot{\nu}, \dot{\mu})$ is more aligned than (ν, μ) if and only if, for all E and F, every EF-extreme experiment is $\dot{EF}$-extreme. For part (ii), Lemma 30(i) implies that if $\nu = \mu$, then $\succsim^A$ is Blackwell monotone for all bets A. Conversely, $\nu \neq \mu$ implies there exists E, F with nonempty strict EF-disagreement region and, hence, a violation of Blackwell monotonicity by Lemma 30(ii).

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