An Axiomatic Model of Persuasion

Alexander M. Jakobsen*

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Abstract

A sender ranks information structures knowing that a receiver processes the information before choosing an action affecting them both. The sender and receiver may differ in their utility functions and/or prior beliefs, yielding a model of dynamic inconsistency when they represent the same individual at two points in time. I take as primitive (i) a collection of preference orderings over all information structures, indexed by menus of acts (the sender's ex-ante preferences for information), and (ii) a collection of correspondences over menus of acts, indexed by signals (the receiver's signal-contingent choice(s) from menus). I provide axiomatic representation theorems characterizing the sender as a sophisticated planner and the receiver as a Bayesian information processor, and show that all parameters can be uniquely identified from the sender's preferences for information. I also establish a series of results characterizing common priors, common utility functions, and intuitive measures of disagreement for these parameters—all in terms of the sender's preferences for information.

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1 Introduction

This paper develops axiomatic foundations for a general model of communication with sender commitment power. As in Kamenica and Gentzkow (2011), the setup involves two agents: Sender and Receiver. By controlling information, Sender attempts to guide Receiver toward actions that are more beneficial to himself. The main result is a representation theorem characterizing Sender as a sophisticated Bayesian planner and Receiver as a Bayesian information processor. Importantly, Sender's *preference for information* is an observable primitive, reflecting the idea that he controls the information (not the actions) available to Receiver.

Sender and Receiver are expected utility maximizers but may differ in their utility functions or prior beliefs. This enables two interpretations of the model. In the *persuasion* interpretation, Sender and Receiver represent distinct individuals, as in "Bayesian persuasion" models (Kamenica and Gentzkow, 2011). In the *behavioral* interpretation, Sender and Receiver represent the same individual at two points in time, yielding a model of dynamically inconsistent behavior. As is well-known, sophisticated, dynamically inconsistent individuals value commitment power (Strotz, 1955). Here, Sender lacks hard commitment power in that he cannot restrict the set of actions available to Receiver (his future self). Instead, he commits to revealing the signal generated by the chosen information structure. Hence, informational choice offers an alternative form of commitment power, and preferences for information reflect preferences for commitment.

To illustrate the main ideas, as well as the behavioral interpretation, consider an individual who must decide whether to consume a dessert (action D) or not (action $\neg D$). An ingredient in the dessert is either unhealthy (state G) or very unhealthy (state B). In period 1, before the decision is to be made, the individual is health-conscious: he prefers not to consume the dessert regardless of the state (preferences v below). He recognizes, however, that he may succumb to temptation when confronted with the choice: his future-self prefers to consume the dessert in state G but to refrain in state B (preferences u).[1]

Lacking hard commitment power, the period-1 individual (Sender) attempts to influence future choice through careful exposure to information. For example, he may consult a specialist who reveals the true state, or browse web sites containing imperfect information about the state. If he acquires sufficient evidence of state B, his period-2 self (Receiver) will refrain from consuming the dessert despite the lack of hard commitment power.

Differences between first- and second-period utility functions induce non-trivial preferences for information. If, for example, both selves assign prior probability 2/3 to state G,

[1] This payoff structure is isomorphic to that of the leading example in Kamenica and Gentzkow (2011), although the interpretation is different.

	G	B
D	0	-2
$\neg D$	1	1

(a) Utilities v

	G	B
D	2	0
$\neg D$	1	1

(b) Utilities u

then Sender prefers perfect information over no information: perfect information results in choice $\neg D$ with probability 1/3, while no information results in choice D with probability 1. However, perfect information is not ideal from Sender's perspective. Consider the following information structure, denoted σ:

	s	t
G	1/4	3/4
B	1	0

This information structure generates signal s in state B, while in state G it generates s with probability 1/4 and t with probability 3/4. Under Bayesian updating, Receiver chooses D at signal t and $\neg D$ at s. Thus, Sender achieves a higher expected payoff from σ than from perfect information, so that his preference for information violates the Blackwell (1951, 1953) information ordering. Similarly, non-common priors also lead to violations of the Blackwell ordering.[2] A key finding of this paper is that such violations are very informative and that, in fact, Sender's preferences for information fully reveal the priors and utilities of both agents.

In the representation, Receiver selects among *acts* (Anscombe and Aumann, 1963): profiles $f = (f_\omega)_{\omega\in\Omega}$ assigning lotteries $f_\omega \in \Delta X$ to states $\omega \in \Omega$, where X and Ω are finite sets of outcomes and states, respectively. Information structures take the form of Blackwell experiments which, as illustrated above, are matrices σ where each column represents a signal and each row ω represents a state-contingent probability distribution over the signals.

Receiver's choices are summarized by a family of *signal-contingent choice correspondences* c^s. A *signal* is a profile $s = (s_\omega)_{\omega\in\Omega}$ of entries from $[0, 1]$ with at least one non-zero entry. In other words, s coincides with a column from some experiment and the entries of s represent likelihoods of the signal being generated in different states of the world. For a signal s and *menu* A (a finite set of acts), $c^s(A) \subseteq A$ is the set of acts chosen by Receiver after observing s. In the representation, choices c^s are rationalized by expected utility maximization under a utility index u, prior μ (full support), and Bayesian updating. The key axiom, *Bayesian Consistency*, expresses an equivalence between scaling probabilities of outcomes and signal realizations, ensuring that Receiver is a Bayesian updater.

[2]Heterogeneous priors can be interpreted as a different source of temptation. In this example, both decision makers could hold utility function u while the second-period prior is skewed in favor of state G. Thus, the effect of temptation is to become biased or delusional in favor of state G, making the dessert seem more attractive. The decision maker knows himself well enough to anticipate this behavior.

Sender's preferences are summarized by a family of preference relations $\succsim^A$ indexed by menus A. Each $\succsim^A$ is an ordering of the set of all Blackwell experiments and represents Sender's *preference for information* when Receiver must choose from A. The statement $\sigma \succsim^A \sigma'$ means Sender would rather expose Receiver to information σ than information σ', given that his outcome is determined by Receiver's signal-contingent choices from A.

In the representation, each $\succsim^A$ ranks experiments according to their expected utility under prior ν (full support), utility index v, and correct forecasting of Receiver's choices. Let $f^s(A)$ denote the act chosen by Receiver from A when signal $s \in \sigma$ realizes.[3] Then

$$V^A(\sigma) := \sum_{\omega \in \Omega} \nu_\omega \sum_{s \in \sigma} s_\omega v(f^s_\omega(A)) \tag{1}$$

is Sender's *value of information* σ *at menu* A, where ν_ω is Sender's prior probability of state ω and $v : X \to \mathbb{R}$ his utility index.[4] This is analogous to an indirect utility function for the sender in Bayesian Persuasion models (Kamenica and Gentzkow, 2011).

The axioms characterizing representation (1) employ both informational preferences $\succsim^A$ and signal-contingent choices c^s. Familiar Independence and Continuity axioms are defined using an appropriate mixture operation on the space of experiments, and the Anscombe-Aumman State Independence axiom is expressed using both preferences $\succsim^A$ and choices c^s. The key axiom, *Consistency*, places minimal restrictions on how preferences $\succsim^A$ may vary as the menu A changes. In particular, it states that Sender only cares about the state-contingent distributions over outcomes generated by the information structure and Receiver's choices, thus ensuring that Sender's prior ν and utility index v are not menu-dependent.

The combined representation theorem (Theorem 3) establishes uniqueness of all parameters (ν, v, μ, and u) given both Sender's preferences for information and Receiver's signal-contingent choices. It turns out, however, that all parameters can be identified from Sender's preferences for information (Theorem 4). Section 4 describes the steps required to elicit these parameters; the full proof is in the Supplementary Appendix.

Sender's preferences can also be used to compare the attributes of Sender and Receiver. I show in Section 5 that Sender and Receiver's utility functions become more aligned as perfect information becomes more attractive to Sender in a class of menus called *bets*.[5] In the limit, when their utility functions coincide, perfect information is Sender's most-preferred information structure in all bets. Similarly, priors are more aligned when more

[3]The statement '$s \in \sigma$' means s is a column of σ (s_ω is the column's entry for row ω). If $c^s(A)$ is single-valued, then $f^s(A)$ is the sole member of $c^s(A)$; otherwise, $f^s(A)$ is a member of the convex hull of $c^s(A)$ and can be interpreted as a distribution (Sender's beliefs) over $c^s(A)$. See section 3.

[4]Abusing notation slightly, let $v(p) := \sum_x v(x)p(x)$ for lotteries $p \in \Delta X$.

[5]A bet is a menu $A = \{f, g\}$ where there exist lotteries $p, q \in \Delta X$ such that $f_\omega, g_\omega \in \{p, q\}$ for all ω.

signals result in agreement regarding the posterior ranking of arbitrary events. "Extreme" experiments are composed of such signals and make Sender's preferences locally monotone with respect to the Blackwell ordering. In the limit, priors coincide and Sender's preferences are globally monotone with respect to the Blackwell ordering in all bets. Finally, the limit cases are combined to establish that Sender and Receiver share both a common prior and utility function if and only if Sender's preferences satisfy the Blackwell ordering in all menus (or even just all bets). Thus, in the behavioral interpretation of the model, dynamically consistent behavior is characterized by adherence to the Blackwell information ordering.

These results illustrate the power and applicability of information structures as objects of choice. Preferences for information may seem rather abstract, but it is not difficult to see how individuals reveal such preferences in different environments. For example, many online newspapers allow subscribers to customize their news feeds by selecting categories (sports, finance, politics, etc.) about which they will be informed of new developments, while online retailers enable custom tailoring of information about new products or services. By customizing such news feeds, individuals reveal what type of information they consider to be the most valuable. Laboratory settings, of course, provide another setting where preferences for information can be directly elicited. This paper does not carry out any empirical or experimental exercises, but demonstrates that informational choice may be a valuable resource for analysts interested in testing models or identifying parameters.

Finally, note that preferences for information are a natural primitive in both interpretations of the model. In Bayesian Persuasion settings, Sender's informational preferences, together with Receiver's signal-contingent choices, are the most an analyst can hope to observe. In the behavioral interpretation, informational choice offers an effective form of commitment power. Hard commitment opportunities are relatively rare compared to the abundance of available information sources. Thus, while an individual might not be able to avoid encountering tempting alternatives, he may be able to resist temptation when it arrives by selectively paying attention to information sources—in particular, ones that are more likely to make tempting alternatives seem less appealing.

1.1 Related Literature

This paper is related to the growing literature on information disclosure with sender commitment power initiated by Kamenica and Gentzkow (2011) (henceforth KG) and Rayo and Segal (2010).[6] My model is most closely related to the framework of KG, where a sender chooses an experiment and a receiver takes an action after observing a signal generated by

[6]In contrast, cheap talk models (Crawford and Sobel, 1982) assume no commitment power.

the experiment. Building on techniques of Aumann and Maschler (1995), KG study when and how the sender can improve his own expected payoff through "persuasion": choosing an experiment and committing to revealing its signal.

My analysis and motivation differs from that of KG in several ways. Rather than studying when the sender might benefit from persuasion, I examine how observed choices can be used to test whether the sender and receiver conform to the KG framework. My representation characterizes what it means for the receiver to be a Bayesian information processor and the sender a sophisticated planner. The characterization is expressed in terms of the choices agents actually make in the KG framework—the sender chooses information, and the receiver chooses among risky alternatives. I also show how observed choices can be used to identify and compare the beliefs and utilities of the agents. While KG take as given a fixed set of actions and consider a sender who is free to choose his most-preferred information structure, my analysis involves a rich set of choice data: for each menu of acts, the sender's full ranking of information structures is observed. The full ranking is needed to characterize the agents and identify parameters. Finally, while KG assume common priors, my framework permits the sender and receiver to hold different priors.[7]

The behavioral interpretation of my model offers a different perspective on temptation and commitment. Starting with Kreps (1979), preferences for flexibility or commitment are typically modeled as preferences over menus of alternatives. Such preferences represent hard commitment in that menus indicate the options available for later consumption. Utilizing preferences over menus of lotteries, Dekel, Lipman, and Rustichini (2001) characterize a representation driven by a set of subject states, extending the representation of Kreps (1979). In a similar setting, Gul and Pesendorfer (2001) characterize a representation where an agent faces temptation and suffers a cost of self-control. Gul and Pesendorfer's model nests a generalization of Strotz (1955). My representation can be interpreted as a Strotz model where the agent controls information, but not the actions, available to his future self.

Behavioral economists have developed models where agents regulate behavior through information suppression or self-signaling. Carrillo and Mariotti (2000) show that, in a model of personal equilibrium, time-inconsistent agents may benefit from acquiring less information, while Grant, Kajii, and Polak (2000) examine when a dynamically consistent individual with non-expected utility preferences prefers more information to less. Benabou and Tirole (2002, 2006) study equilibrium models where players rationally limit information available to future selves. In the persuasion literature, Lipnowski and Mathevet (2018) examine how a benevolent principal should disclose information to agents who are susceptible to

[7] Alonso and Câmara (2016) extend the KG framework to allow heterogeneous priors and find that (generically) the sender benefits from persuasion under heterogeneous priors.

temptation, reference-dependence, or other behavioral phenomena. Similarly, the behavioral interpretation of my model provides a general analysis of the incentives for information acquisition for individuals lacking time-consistent preferences or prior beliefs.

Azrieli and Lehrer (2008) consider preferences over information structures and provide necessary and sufficient conditions for such a preference to be represented by expected utility in some decision problem.[8] In their representation, with the prior is taken as given, a utility index and menu of actions are deduced from the preference for information but cannot be uniquely pinned down. Azrieli and Lehrer (2008) note that their axioms can be modified to allow an endogenous prior but that it, too, cannot be uniquely identified. My model resolves these identification issues by examining preferences for information in all exogenously specified menus—even with time-inconsistent priors or utilities, all parameters can be uniquely identified from this richer collection of preferences.

Several authors have studied Bayesian updating from a decision-theoretic perspective. Ghirardato (2002) develops a representation using conditional preferences over acts; that is, families of preferences indexed by events, with the interpretation that the event represents an observed signal. Karni (2007) uses a similar family of conditional preferences defined over conditional acts. The extra structure of conditional acts permits both prior beliefs and state-dependent utilities to be identified, in addition to testing Bayesian updating of partitional information. Wang (2003) axiomatizes Bayes' rule and some of its extensions in a setting with conditional preferences over infinite-horizon consumption-information profiles; preferences are conditioned on sequences of previously realized events. My representation characterizes Bayesian updating using signal-contingent preferences over standard Anscombe-Aumann acts. The set of signals is richer than the state space over which acts are defined, enabling a simple and intuitive characterization.

Finally, Lu (2016) shows how random choice data reveals an individual's information, provided the individual is a Bayesian expected utility maximizer. Decision-theoretic models of rational inattention[9] also use standard choice primitives to make inferences about an individual's preferences, beliefs, and information processing ability. My framework uses informational choice to make inferences about one's underlying tastes and beliefs.

[8]See also Gilboa and Lehrer (1991), who study a similar problem for partitional information structures.

[9]See Denti, Mihm, de Oliveira, and Ozbek (2016), Ellis (2018), and Caplin and Dean (2015)

2 Framework and Notation

2.1 Outcomes, lotteries, acts

Let X denote a finite set of $N \geq 2$ *outcomes*, with generic members denoted x, y. Elements of ΔX, *lotteries*, are denoted p, q.[10] A lottery p assigns probability $p(x)$ to outcome x.

A *utility index* is a function $u : X \to \mathbb{R}$. If $p \in \Delta X$ and u is a utility index, let $u(p) := \sum_{x \in X} u(x)p(x)$ denote the expected utility of p. The notation $u' \approx u$ indicates that u' is a positive affine transformation of u.

Let Ω denote a finite, exogenous set of $W \geq 2$ states. Arbitrary states are typically denoted ω, ω', while members of $\Delta\Omega$, *probability distributions* over Ω, are denoted μ or ν. As a notational convention, subscripts denote states. For example, a distribution $\mu \in \Delta\Omega$ may be expressed as $\mu = (\mu_\omega)_{\omega\in\Omega}$, where μ_ω is the probability assigned to state ω.

A function $f : \Omega \to \Delta X$ is an Anscombe-Aumann *act*. Let F denote the set of all acts. Acts are typically denoted f, g, h, and may be written as profiles: $f = (f_\omega)_{\omega\in\Omega}$, where $f_\omega \in \Delta X$. The set F is equipped with the standard mixing operation: if $f, g \in F$ and $\alpha \in [0, 1]$, then $\alpha f + (1 - \alpha)g := (\alpha f_\omega + (1 - \alpha)g_\omega)_{\omega\in\Omega}$.

A *menu* is a finite, nonempty set of acts. Menus are typically denoted A, B. Let $\mathcal{A}$ denote the set of all menus. Both F and ΔX are endowed with the standard Euclidean metric, and $\mathcal{A}$ with the associated Hausdorff metric. I use $N(\cdot)$ to denote open neighborhoods, or $N^\varepsilon(\cdot)$ when specifying the radius of the neighborhood. For example, $N^\varepsilon(p)$ is an open ball of radius ε around lottery p.

2.2 Blackwell Experiments

Definition 1 (Blackwell Experiment)**.** A matrix σ with entries in $[0, 1]$ is a (finite) *Blackwell experiment* if it has exactly W rows, no columns consisting only of zeros and, for each row, the sum of entries is exactly one. Let $\mathcal{E}$ denote the set of all Blackwell experiments.

Each column of σ represents a signal that might be generated, and each row a state-contingent probability distribution over such signals. The requirement that each column contains at least one nonzero entry eliminates signals that have zero probability of occurrence in each state. Note that entries in any given column are not required to sum to one.

It will be convenient to express experiments in terms of their columns. Let

$$S := \{s = (s_\omega)_{\omega\in\Omega} \in [0, 1]^\Omega : \exists\omega \text{ such that } s_\omega \neq 0\}. \tag{2}$$

[10] For any finite set Y, ΔY denotes the standard probability simplex over Y, equipped with the usual convex mixture operation.

Elements of S are called *signals*. Clearly, every column of an experiment σ corresponds to a signal s where s_ω is the entry for the column in row ω.[11]

The statement '$s \in \sigma$' means s is a column of σ. Note that an experiment may have duplicate columns. When quantifying over signals in an experiment, different columns of σ are distinguished even if they are duplicates. For example, the requirement that each row in σ has entries summing to one may be expressed as '$\forall\omega$, $\sum_{s\in\sigma} s_\omega = 1$' because the summation notation implicitly distinguishes between duplicate columns of σ. Similarly, statements like '$\forall s \in \sigma$, $y^s \in Y$' associate potentially different members of Y to different columns of σ, even if those columns are duplicates.

If $\sigma \in \mathcal{E}$ and $\alpha \in (0,1)$, let $\alpha\sigma$ denote the matrix formed by multiplying each entry of σ by α. If $\sigma, \sigma' \in \mathcal{E}$ and $\alpha \in (0,1)$, then $\alpha\sigma \cup (1-\alpha)\sigma'$ denotes the matrix formed by appending $(1-\alpha)\sigma'$ to the right of $\alpha\sigma$. It is easy to verify that this mixture yields a well-defined experiment, although the operation is not commutative. Intuitively, $\alpha\sigma \cup (1-\alpha)\sigma'$ is the information structure coming about if nature randomly selects between σ and σ' (with probabilities α and $1-\alpha$) before using the chosen matrix to generate a signal. To the best of my knowledge, this paper is the first to employ this mixture operation.

An additional mixture operation on acts can be defined using the notation of signals and experiments. If $\sigma \in \mathcal{E}$ and $f^s \in F$ for every $s \in \sigma$, let $\sum_{s\in\sigma} sf^s := \left(\sum_{s\in\sigma} s_\omega f^s_\omega\right)_{\omega\in\Omega}$. This is the act formed by applying weight s_ω to f^s_ω ($s \in \sigma$) in state ω. A special case is the operation $sf+(1-s)g := (s_\omega f_\omega + (1-s_\omega)g_\omega)_{\omega\in\Omega}$, corresponding to $\sigma = [s, 1-s]$, $f^s = f$, and $f^{1-s} = g$.

Finally, signals are endowed with the standard Euclidean metric. Interpreting an experiment as a *set* of columns (signals), $\mathcal{E}$ is endowed with the associated Hausdorff metric.[12] Open neighborhoods are denoted by $N(\cdot)$ or $N^\varepsilon(\cdot)$; for example, $N^\varepsilon(s)$ is an open ball of radius ε around signal s.

2.3 Primitives

I take as primitive two sets of choice data:

(1) For each menu $A \in \mathcal{A}$, a preference $\succsim^A$ over $\mathcal{E}$. Let $\succsim = (\succsim^A)_{A\in\mathcal{A}}$.

(2) For each signal $s \in S$, a choice correspondence c^s such that, for each menu A, $c^s(A)$ is a nonempty subset of A. Let $c = (c^s)_{s\in S}$.

[11] Note that a signal refers to both a particular message that might be generated by an experiment as well as the state-contingent likelihoods of that message. These likelihoods, together with a prior, are all that is needed to compute a Bayesian posterior.

[12] Technically, this makes the Hausdorff metric on $\mathcal{E}$ a pseudometric because distinct experiments may have distance zero. For example, reordering the columns of σ results in a (technically) distinct experiment σ', although the Hausdorff distance between σ and σ' is zero.

The family $\succsim = (\succsim^A)_{A\in\mathcal{A}}$ captures Sender's preferences for information. In particular, $\sigma \succsim^A \sigma'$ means Sender prefers to expose Receiver to information σ rather than σ' given that Receiver chooses from A after observing a signal.

Receiver's choices are captured by the collection $c = (c^s)_{s\in S}$. For each signal s and menu A, $c^s(A)$ contains the acts chosen by Receiver from A after observing signal s. In practice, Receiver's choice is conditioned on a pair (σ, s) where $s \in \sigma$; that is, a signal must be generated by some experiment σ. However, for a Bayesian information processor, only the entries of s (not the other columns of σ) matter. To minimize notation, I condition choices on signals s instead of pairs (σ, s).[13]

3 The Representation

3.1 Value of Information and Bayesian Representations

The goal is to represent Sender as a sophisticated Bayesian planner and Receiver a Bayesian information processor. In the representation, each agent is an expected utility maximizer, but Sender and Receiver need not have a common prior or utility function.

A *selection* is a mapping $(s, A) \mapsto f^s(A)$ such that, for all $s \in S$ and $A \in \mathcal{A}$, $f^s(A) \in \Delta c^s(A)$; the selection is *stable* if for every $A \in \mathcal{A}$ there is a sequence $(A^n)_{n=1}^{\infty}$ of menus such that, for all $s \in S$, $c^s(A^n) \to f^s(A)$ in the Hausdorff metric.

Definition 2 (Value of Information Representation)**.** A pair (ν, v) and a stable selection $(s, A) \mapsto f^s(A)$ constitute a *Value of Information Representation* for $(\succsim, c)$ if $\nu \in \Delta\Omega$ has full support, $v : X \to \mathbb{R}$ is non-constant and, for each menu A, the function

$$V^A(\sigma) := \sum_{\omega\in\Omega} \nu_\omega \sum_{s\in\sigma} s_\omega v(f^s_\omega(A)) \tag{3}$$

represents $\succsim^A$.

In a Value of Information Representation, Sender correctly forecasts the signal-contingent choices made by Receiver and assigns expected utility to experiments σ using utility function v and subjective prior ν. If $c^s(A)$ is a singleton, $f^s(A)$ coincides with the sole member of $c^s(A)$; otherwise, $f^s(A)$ is a member of $\Delta c^s(A)$, representing a probability distribution over the acts in $c^s(A)$. Thus, the selection $(s, A) \mapsto f^s(A)$ may be interpreted as Sender's beliefs about Receiver's choices from A at signal s, and $V^A(\sigma)$ as Sender's expected utility from committing to σ given his beliefs.

[13] If choices were conditioned on pairs (σ, s), the following additional axiom would be required: for all σ, σ' with $s \in \sigma$ and $s \in \sigma'$, $c^{(\sigma,s)} = c^{(\sigma',s)}$.

A special case involves *sender-preferred tie-breaking.* Under this selection, Sender behaves as if Receiver breaks ties in Sender's favor, so that $\succsim^A$ is represented by the function

$$\overline{V}^A(\sigma) := \max \sum_{\omega \in \Omega} \nu_\omega \sum_{s \in \sigma} s_\omega v(f^s_\omega) \quad \text{subject to} \quad f^s \in c^s(A). \tag{4}$$

Sender-preferred tie-breaking is analytically convenient and a standard assumption in the literature, but the results of this paper are robust to the more general class of representations utilizing stable selections. As explained in sections 4 and 5 below, sender-preferred tie-breaking simplifies some of the subsequent analysis of the model.

Although the definition and interpretation of a Value of Information representation does not involve any assumptions regarding Receiver's behavior—any signal-contingent choice correspondences $(c^s)_{s \in S}$ will suffice—the standard model involves a Bayesian receiver:

Definition 3 (Bayesian Representation)**.** A pair (μ, u) is a *Bayesian Representation* for c if $\mu \in \Delta\Omega$ has full support, $u : X \to \mathbb{R}$ is non-constant and, for all $s \in S$ and $A \in \mathcal{A}$,

$$c^s(A) = \left\{ f \in A \ : \ \forall g \in A, \ \sum_\omega u(f_\omega)\mu^s_\omega \geq \sum_\omega u(g_\omega)\mu^s_\omega \right\} \tag{5}$$

where the posteriors μ^s satisfy Bayes' rule:

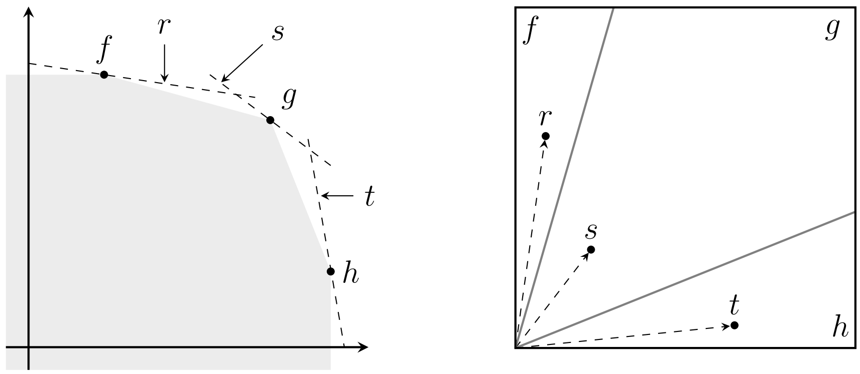
$$\forall \omega \in \Omega, \ \mu^s_\omega = \frac{\mu_\omega s_\omega}{\sum_{\omega' \in \Omega} \mu_{\omega'} s_{\omega'}}. \tag{6}$$

In a Bayesian Representation, each correspondence c^s is rationalized by expected utility maximization with prior μ, utility index u, and Bayesian updating: upon observing signal s, Receiver updates his prior μ to the Bayesian posterior μ^s given by (6), then chooses $f \in A$ if and only if f maximizes expected utility under beliefs μ^s. Figure 1 provides geometric representations of this behavior.

3.2 Characterization of Sender

The first set of axioms express seven properties of Sender's behavior. Of these, the first three are essentially the von Neumann-Morgenstern axioms, adapted to operate on experiments and their mixtures.

Axiom 1.1 (Rationality). Each $\succsim^A$ is complete and transitive.



(a) Utility Space (b) Signal space

Figure 1: Geometric representations of Receiver's behavior when $|\Omega| = 2$. Receiver prefers f over g at signal $\tilde{s}$ if and only if $\sum_\omega u(f_\omega)\tilde{s}_\omega\mu_\omega \geq \sum_\omega u(g_\omega)\tilde{s}_\omega\mu_\omega$. Thus, in utility space, acts f correspond to points $(\mu_1 u(f_1), \mu_2 u(f_2))$, and choices at signals $\tilde{s}$ are determined by the ratio $\tilde{s}_1/\tilde{s}_2$. Consequently, Receiver's choices from $A = \{f, g, h\}$ partition S into convex cones. Arrows pointing to signals in (b) are perpendicular to the corresponding lines in (a).

Axiom 1.2 (Independence). If $\sigma \succ^A \sigma'$ and $\alpha \in (0,1)$, then $\alpha\sigma \cup (1-\alpha)\sigma'' \succ^A \alpha\sigma' \cup (1-\alpha)\sigma''$ for all σ''.

Axiom 1.3 (Continuity). If $\sigma \succ^A \sigma' \succ^A \sigma''$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha\sigma \cup (1-\alpha)\sigma'' \succ^A \sigma' \succ^A \beta\sigma \cup (1-\beta)\sigma''$.

For each menu A, let $\mathcal{E}^c(A)$ denote the set of all experiments σ such that $c^s(A)$ is single-valued for all $s \in \sigma$. The next axiom is needed to disentangle Sender's beliefs and utilities.

Axiom 1.4 (Non-Degeneracy). There exist $A \in \mathcal{A}$ and $\sigma, \sigma' \in \mathcal{E}^c(A)$ such that $\sigma \succ^A \sigma'$.

If $\sigma \in \mathcal{E}^c(A)$, let $c^s(A) \in F$ denote the unique act chosen by Receiver at s, and $c^s_\omega(A) \in \Delta X$ the lottery specified by $c^s(A)$ in state ω. Then $c^\sigma(A) := \sum_{s\in\sigma} s c^s(A) \in F$ is the *induced act* for σ at A. When using the notation $c^\sigma(A)$, it is implicit that $\sigma \in \mathcal{E}^c(A)$. Intuitively, $c^\sigma(A)$ is the "average" act chosen by Receiver when signals are generated by σ. It is formed by applying weight s_ω to $c^s_\omega(A)$ in state ω, so that the state-contingent distributions over signals given by σ yield state-contingent mixtures of lotteries. Each state-contingent mixture of lotteries is reduced to a single lottery, yielding an Anscombe-Aumann act.

Axiom 1.5 (Consistency). If $c^\sigma(A) = c^{\hat{\sigma}}(B)$, $c^{\sigma'}(A) = c^{\hat{\sigma}'}(B)$, and $\sigma \succsim^A \sigma'$, then $\hat{\sigma} \succsim^B \hat{\sigma}'$.

Consistency states that Sender's preferences only depend on the induced act, provided the induced act exists. Thus, Sender correctly forecasts Receiver's choices and cares only about the distribution of outcomes in each state of the world. As a standard Bayesian, Sender reduces each state-contingent mixture of lotteries to a single lottery. Consequently, Sender's preferences do not depend on which particular combination of menu and experiment give rise to an induced act.

For $p \in \Delta X$, $h \in F$, and $\omega \in \Omega$, let $p[\omega]h$ denote the act formed by taking h and replacing h_ω with p. The next axiom is analogous to the State Independence axiom in the Anscombe-Aumann model, once again adapted to operate on experiments.[14]

Axiom 1.6 (State Independence). Suppose $c^{\sigma}(A) = p[\omega]h$ and $c^{\sigma'}(A) = q[\omega]h$ while $c^{\hat{\sigma}}(A) = p[\omega']\hat{h}$ and $c^{\hat{\sigma}'}(A) = q[\omega']\hat{h}$. Then $\sigma \succsim^A \sigma'$ implies $\hat{\sigma} \succsim^A \hat{\sigma}'$.

Combined, Axioms 1.1–1.6 describe Sender as a standard, sophisticated Bayesian who ranks information structures according to their associated distributions of Receiver's choices (induced acts). Sender's implied ranking of induced acts satisfies the Anscombe-Aumann axioms, but induced acts are only defined for experiments where every signal realization results in a unique choice by the Receiver. To accommodate situations involving "ties" (multi-valued $c^s(A)$ for some $s \in \sigma$), one additional axiom is required.

A sequence of menus $(A^n)_{n=1}^{\infty}$ *converges in choice to* A, denoted $A^n \to^c A$, if there is a set $A \subseteq B \subseteq \text{co}(A)$ such that $A^n \to B$ and, for all s, $c^s(A^n)$ converges to a singleton in the Hausdorff metric.[15] If $A^n \to^c A$ and $s \in S$, then $c^s(A^\infty) := \lim c^s(A^n)$ is well-defined, as is $c^\sigma(A^\infty) := \sum_{s \in \sigma} c^s(A^\infty)$ for all σ. Finally, if $A^n \to^c A$, then $\succsim^{A^n} \to^c \succsim^A$ if, for all $\sigma, \sigma' \in \mathcal{E}$, $\sigma \succsim^A \sigma'$ if and only if there exists $\hat{A} \in \mathcal{A}$, $\hat{\sigma}, \hat{\sigma}', \sigma^* \in \mathcal{E}^c(\hat{A})$ and $\alpha \in (0,1)$ such that $c^{\hat{\sigma}}(\hat{A}) = \alpha c^\sigma(A^\infty) + (1-\alpha)c^{\sigma^*}(\hat{A})$, $c^{\hat{\sigma}'}(\hat{A}) = \alpha c^{\sigma'}(A^\infty) + (1-\alpha)c^{\sigma^*}(\hat{A})$, and $\hat{\sigma} \succsim^{\hat{A}} \hat{\sigma}'$.

Axiom 1.7 (Stability). For every A, there exists $A^n \to^c A$ such that $\succsim^{A^n} \to^c \succsim^A$.

Stability expresses the idea that Sender's preferences at A can be approximated by those in a sequence of nearby menus where Receiver's choices are single-valued in the limit. In particular, $\succsim^{A^n} \to^c \succsim^A$ means that at every $s \in S$, Receiver's choice from A^n converges to a point in the convex hull of A, and Sender's preferences between σ and σ' are determined by the limit induced acts $c^\sigma(A^\infty)$ and $c^{\sigma'}(A^\infty)$, the ranking of which is revealed by some menu $\hat{A}$. Thus, it is as if Sender believes Receiver's choice at s is some act in the convex hull of

[14] The standard axiom says: if ω, ω' are non-null and $p[\omega]h$ is weakly preferred over $q[\omega]h$, then $p[\omega']\hat{h}$ is weakly preferred over $q[\omega']\hat{h}$ for all $\hat{h}$. Axiom 1.6 rules out null states, so that ν has full support.

[15] For any menu A, let $\text{co}(A)$ denote the convex hull of A.

$c^s(A)$, which can be interpreted as a probability distribution, or beliefs, over acts in $c^s(A)$.

Theorem 1. *Suppose c has a Bayesian Representation. Then $(\succsim, c)$ satisfies Axioms 1.1–1.7 if and only if it has a Value of Information Representation. Moreover, ν is unique and v is unique up to positive affine transformation.*

Theorem 1 is an intermediate step toward a complete representation theorem. In particular, it states that Axioms 1.1–1.7 are necessary and sufficient for existence of a Value of Information Representation provided c has a Bayesian Representation. The proof of Theorem 1 (as well as that of Theorem 2 below) is in the main appendix; all other proofs are in the Online Appendix.

Although most of Axioms 1.1–1.7 are straightforward adaptations of the Anscombe-Aumann axioms, Theorem 1 is not a direct corollary of the Anscombe-Aumann theorem. There are two obstacles. First, it is not obvious that variation in σ induces enough variation in Receiver's choices to establish existence of an expected utility representation for Sender, or uniqueness of ν and v provided a representation exists. A key step of the proof constructs a menu A^* from which existence is established and candidates for ν and v can be identified. Second, it is also not obvious that the Consistency axiom is strong enough to ensure menu-independent beliefs and utilities can be obtained for Sender. If two menus give rise to disjoint sets of induced acts, then Consistency seemingly has no bite and Sender could hold different beliefs and/or utilities in those menus. The main challenge of the proof is to show that any two menus can be connected by a finite sequence of menus with significantly overlapping sets of induced acts along the way, thus ensuring uniqueness.

Finally, note that the only role of Axiom 1.7 is to deal with ties. In fact, if Receiver is Bayesian, Axioms 1.1–1.6 are sufficient to establish a unique ν and v so that, for every A, the restriction of $\succsim^A$ to $\mathcal{E}^c(A)$ has the desired Value of Information Representation. This holds because ties are highly "non-generic" when Receiver is Bayesian: if some $c^s(A)$ is multi-valued, then arbitrarily small perturbations of A (or, typically, of s) eliminate the tie. In the appendix, I show that replacing Axiom 1.7 with the following axiom characterizes sender-preferred tie-breaking:

Axiom 1.7' (Sender Optimism). For all $\sigma, \sigma' \in \mathcal{E}$ and $A \in \mathcal{A}$, $\sigma \succsim^A \sigma'$ if and only if for all $A^n \to^c A$, there exists $B^n \to^c A$, $\hat{A} \in \mathcal{A}$, $\hat{\sigma}, \hat{\sigma}', \sigma^* \in \mathcal{E}^c(\hat{A})$ and $\alpha \in (0,1)$ such that $c^{\hat{\sigma}}(\hat{A}) = \alpha c^{\sigma}(B^\infty) + (1-\alpha)c^{\sigma^*}(\hat{A})$, $c^{\hat{\sigma}'}(\hat{A}) = \alpha c^{\sigma'}(A^\infty) + (1-\alpha)c^{\sigma^*}(\hat{A})$, and $\hat{\sigma} \succsim^{\hat{A}} \hat{\sigma}'$.

This axiom states that $\sigma \succsim^A \sigma'$ if, for every possible limit induced act $g = c^{\sigma'}(A^\infty)$, there is a limit induced act $c^\sigma(B^\infty)$ that is revealed-preferred to g. That is, the best-case scenario for σ' at A is inferior to that of σ.

3.3 Characterization of Receiver

The final two axioms characterize Receiver as a standard Bayesian; unlike the first set of axioms, these involve only Receiver choice data c. If $\alpha \in [0,1]$ and $A, B \in \mathcal{A}$, then $\alpha A + (1-\alpha)B := \{\alpha f + (1-\alpha)g : f \in A,\ g \in B\}$. If $L \subseteq \Delta X$ is finite, $h \in F$, and $\omega \in \Omega$, then $L[\omega]h := \{p[\omega]h : p \in L\}$.

Axiom 2.1 (Standard Receiver Preferences).

(i) Each c^s satisfies WARP: if $f, g \in A \cap B$, $f \in c^s(A)$, and $g \in c^s(B)$, then $f \in c^s(B)$.

(ii) For every s, there exists A such that $c^s(A) \neq A$.

(iii) If $A, B \in \mathcal{A}$, $\alpha \in [0,1]$, and $s \in S$, then $c^s(\alpha A + (1-\alpha)B) \subseteq \alpha c^s(A) + (1-\alpha)c^s(B)$.

(iv) Each c^s is upper-hemicontinuous.

(v) If $L \subseteq \Delta X$ is finite, $h \in F$, and $s_\omega, s'_{\omega'} > 0$, then $c^s_\omega(L[\omega]h) \subseteq c^{s'}_{\omega'}(L[\omega']h')$.

Axiom 2.1 states five properties required for each c^s to be rationalized by subjective expected utility. Parts (i) and (ii) imply each c^s has a non-degenerate rationalizing preference $\succsim^s$, while parts (iii) and (iv) ensure $\succsim^s$ satisfies standard Independence and Continuity properties. Finally, part (v) implies $\succsim^s$ satisfies a version of the State Independence axiom; in particular, Receiver ranks lotteries independently of the state and signal realization.

If $A \in \mathcal{A}$, $s \in S$, and $h \in F$, let $sA + (1-s)h := \{sf + (1-s)h : f \in A\}$.

Axiom 2.2 (Bayesian Consistency). If $tf + (1-t)h \in c^s(tA + (1-t)h)$, then $sf + (1-s)h' \in c^t(sA + (1-s)h')$.

Axiom 2.2 states that signal likelihoods are exchangeable with outcome probabilities. When comparing acts in $tA + (1-t)h$, an expected utility maximizer effectively ignores the $(1-t)h$ component, resulting in a comparison between acts of A whose outcomes have been scaled by signal t. Bayesian Consistency states that making such comparisons after observing s is equivalent to comparing acts in $sA + (1-s)h$ after observing t.

Theorem 2. *The collection $(c^s)_{s \in S}$ satisfies Axioms 2.1 and 2.2 if and only if it has a Bayesian representation (μ, u). Furthermore, μ is unique and u is unique up to positive affine transformation.*

The proof of Theorem 2 is quite simple. By Axiom 2.1, there exists a utility index u and beliefs $\mu^s \in \Delta\Omega$ such that c^s is rationalized by expected utility with beliefs μ^s and utility u.

Axiom 2.2 ensures μ^s is the Bayesian posterior of $\mu := \mu^e$, where $e = (1, \ldots, 1) \in S$ is an uninformative signal; that is, $\mu^s_\omega = \frac{s_\omega \mu^e_\omega}{\sum_{\omega' \in \Omega} s_{\omega'} \mu^e_{\omega'}}$.

3.4 Combined Representation

The following result is an immediate consequence of Theorems 1 and 2.

Theorem 3. *The primitives $(\succsim, c)$ satisfy Axioms 1.1–1.7 and 2.1–2.2 if and only if $(\succsim, c)$ has a Value of Information Representation and c a Bayesian Representation. The priors ν, μ are unique, and the utility indices v, u are unique up to positive affine transformation.*

Proof. First apply Theorem 2 to establish a Bayesian Representation (μ, u) with the desired uniqueness properties. Then apply Theorem 1 to establish a Value of Information Representation, also with the desired uniqueness properties. □

4 Identification

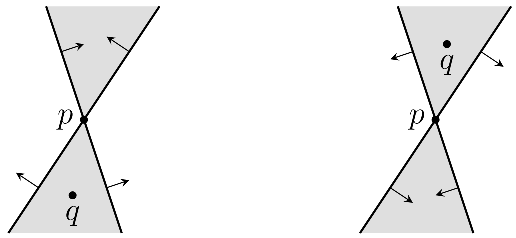
Theorem 3 characterizes the behavior of Sender and Receiver in terms of the primitives $(\succsim, c)$ and shows that ν, μ, v, and u can be identified from those primitives. This section establishes a stronger identification result: all four parameters can be identified using only Sender's preferences $(\succsim^A)_{A \in \mathcal{A}}$.

Definition 4. Sender preferences $\succsim = (\succsim^A)_{A \in \mathcal{A}}$ are *representable by* (ν, μ, v, u) if there exists $c = (c^s)_{s \in S}$ such that $(\succsim, c)$ has a Value of Information Representation with parameters (ν, v) and c has a Bayesian Representation with parameters (μ, u).

Theorem 4. *If $\succsim$ is representable by both (ν, μ, v, u) and (ν', μ', v', u'), then $\nu = \nu'$, $\mu = \mu'$, $v \approx v'$, and $u \approx u'$.*

Theorem 4 states that informational preferences $(\succsim^A)_{A \in \mathcal{A}}$, alone, are sufficient to pin down the priors ν, μ and utility indices v, u. For identification purposes, Receiver's choices $(c^s)_{s \in S}$ are not required. This formally expresses the idea that, for Bayesian decision makers, preferences for information are powerful and revealing primitives of analysis. Note that this result does not depend on Sender's attitude toward, or beliefs about, ties: the prevalence of menu-experiment pairs *not* involving ties pins down the parameters independently of which stable selection is employed.[16]

[16]This result suggests it may be possible characterize the model using only Sender's preferences $\succsim$ as primitive. I do not pursue this possibility here, but consider it an interesting avenue for future research.



(a) (b)

Figure 2: Identifying the agreement region (shaded), from which two linear indifference curves in ΔX can be identified. Panels (a) and (b) indicate the two possibilities for the direction of increasing utilities that are consistent with the agreement region. In each case, lottery p is strictly preferred over q by both agents.

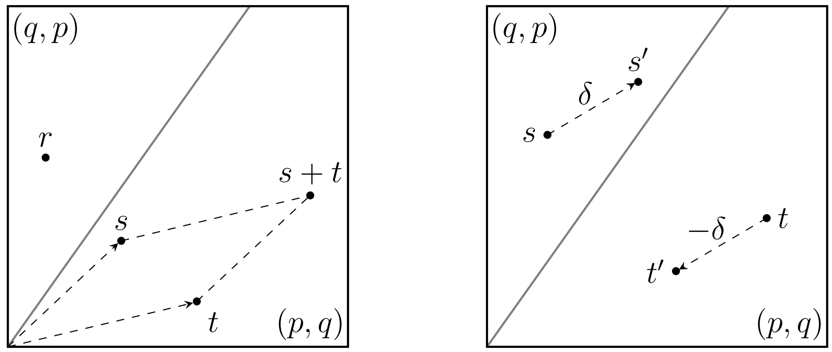
To sketch the proof, some additional notation and terminology is required. A *bet* is a menu of the form $A = \{pEq, pFq\}$ where $E \not\subseteq F$ and $F \not\subseteq E$. Whenever the need to be explicit about E and F arises, I refer to such menus as *EF-bets*. Similarly, a bet may be referred to as a *pq-bet*. Receiver's choices from a bet $A = \{pEq, pFq\}$ depend only on his ranking of p and q and which event has higher posterior probability. For example, if $u(p) > u(q)$, then Receiver chooses pEq after observing s if and only if $\mu^s(E) > \mu^s(F)$. Hence, cardinal properties of u do not influence choice in bets.[17]

For each ω, let $e^\omega \in S$ denote the signal s such that $s_\omega = 1$ and $s_{\omega'} = 0$ for all $\omega' \neq \omega$. Then $\sigma^* := [e^\omega : \omega \in \Omega]$ (the identity matrix) denotes *perfect information*; that is, σ^* reveals the true state ω.

The proof of Theorem 4 involves four steps. First, fix $p, q \in \Delta X$. Suppose there is a neighborhood around q such that $\sigma^* \succsim^{A'} \sigma$ for all $\sigma \in \mathcal{E}$ and all pq'-bets where q' is in the neighborhood. It follows that Sender and Receiver *agree on the ranking of p and q* in that either $[v(p) > v(q)$ and $u(p) > u(q)]$ or $[v(q) > v(p)$ and $u(q) > u(p)]$. By fixing p and varying q, one can identify the set of all q where Sender and Receiver agree on the ranking of p and q. This agreement region (a subset of ΔX) reveals two linear indifference curves. At this point, however, it is not clear which curve corresponds to v and which corresponds to u, or what the direction of increasing utility is (see Figure 2).

The second step is to elicit μ and ν. Let $A = \{pEq, pFq\}$ be a bet where neither agent is indifferent between p and q (such p and q can be found using the agreement region identified in step 1). To identify μ, it will suffice to identify which signals make Receiver rank E strictly more likely than F. The idea is that such *equivalent signals* lead Receiver to make the same choice in any EF-bet, so that merging such signals does not change Sender's value of information. For example, if $s, t \in \sigma$ each result in Receiver choosing pEq, then so does

[17] The restrictions $E \not\subseteq F$ and $F \not\subseteq E$ ensure that neither event dominates the other regardless of which signal Receiver observes. Thus, decision making in bets is responsive to information.



(a) (b)

Figure 3: In (a), s and t are equivalent because they each make Receiver choose (p,q) from $A = \{(p,q),(q,p)\}$. Thus, so does $s+t$, so that $[r,s,t] \sim^A [r,s+t]$. The line separating the (p,q) and (q,p) regions has slope $\frac{\mu_1}{\mu_2}$. Thus, knowing which signals are equivalent reveals μ. In (b), $[s,t] \sim^A [s+\delta, t-\delta]$ implies $\frac{\delta_1}{\delta_2} = \frac{\nu_2}{\nu_1}$, revealing ν.

signal $s+t$. Thus, Sender's ranking of experiments of the form $\sigma = [r,s,t]$ and $\sigma = [r,s+t]$ indicate whether s and t are equivalent, from which μ can be identified (see Figure 3a). To identify ν, once again let $A = \{pEq, pFq\}$ be a bet where neither agent is indifferent between p and q. Knowing μ, one can construct experiments $\sigma = [s,t]$ and $\sigma^\delta = [s+\delta, t-\delta]$ where s and t result in different Receiver choices and $\delta \in \mathbb{R}^\Omega$ is sufficiently small that Receiver's choice at $s+\delta$ coincides with that at s and his choice at $t-\delta$ coincides with that at t. By eliciting δ such that $\sigma \sim^A \sigma^\delta$, one can deduce ν (see Figure 3b).

The third step is to determine which indifference curve from step 1 belongs to Sender and which belongs to Receiver. Knowing the two possibilities, as well as ν and μ, one can construct binary menus $A = \{f,g\}$ that reveal Receiver's indifference curves. For example, with two states of the world, Receiver is indifferent between f and g at s if and only if $s_1\mu_1[u(f_1)-u(g_1)] + s_2\mu_2[u(f_2)-u(g_2)] = 0$. Replacing u with $-u$ would not alter the set of admissible s, but a cardinally different function v would. By examining an analogous concept of "equivalent signals" in such menus, Sender's preferences reveal the set of all s satisfying the identity and, hence, which indifference curve belongs to Receiver. Then, knowing the agreement region from step 1, the indifference curves for Sender are revealed as well.[18]

The previous step results in utility indices (v,u) such that either (v,u) or $(-v,-u)$ are the correct functions (up to positive affine transformation). To determine which pair is correct, the key is to consider a menu $A = \{f,g,h\}$ where, under index u, each act is the unique optimum under some signal. Then, under index $-u$, one act is never selected at any

[18]Under sender-preferred tie-breaking, this step can effectively be skipped because $v(p) = v(q)$ if and only if Sender is indifferent among all information structures in every pq-bet. This immediately reveals Sender's indifference curve in step 1.

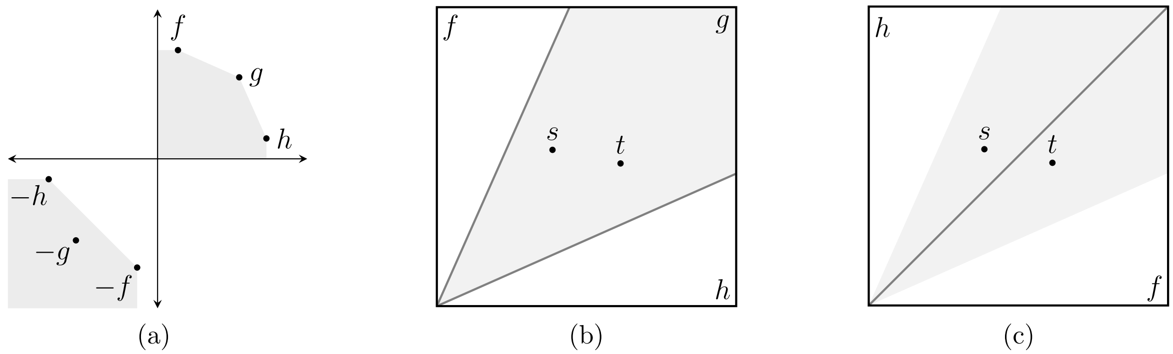


Figure 4: Identifying v and u. If f, g, and h are chosen by Receiver under (v, u), then only f and h are chosen under $(-v, -u)$ (panel (a)). Thus, (v, u) and $(-v, -u)$ yield different divisions of S (panels (b) and (c)). In (b), Sender ranks $\sigma = [s, t]$ indifferent to e because both signals result in choice g. In (c), s and t yield different choices, and therefore (for most choices of f and h) σ is not ranked indifferent to e. Thus, only one of (v, u) or $(-v, -u)$ can be consistent with Sender's preferences $(\succsim^A)_{A \in \mathcal{A}}$.

signal. This yields a different division of S, so that there are experiments σ, σ' that Sender ranks indifferent under $(-v, -u)$ but not under (v, u); see Figure 4.

5 Measures of Consistency

In this section, I show how data $(\succsim, c)$ can be used to make comparisons between the priors and utilities of Sender and Receiver. First, I provide a definition of what it means for the utility indices of Sender and Receiver to be more aligned. I characterize this relation—as well as the limit case $v \approx u$—in terms of Sender's preference for information. Then, I conduct a similar exercise for the priors: I define what it means for ν and μ to be more aligned and characterize this both this relation and the limit case $\nu = \mu$ in terms of Sender's preference for information.

Several results involve the Blackwell information ordering, denoted $\sqsupseteq$. This is a partial order on $\mathcal{E}$ where $\sigma \sqsupseteq \sigma'$ if and only if σ' is a *garbling* of σ; that is, $\sigma' = \sigma M$, where M is a stochastic matrix (each row of M is a probability distribution).[19] Clearly, $\sigma^* \sqsupseteq \sigma \sqsupseteq e$ for all σ, where σ^* denotes perfect information and $e = (1, \dots, 1) \in S$ denotes *no information*.

Definition 5. Primitives $(\succsim, c)$ are *representable by* (ν, μ, v, u) if $\succsim$ has a Value of Information Representation with parameters (ν, v) and c has a Bayesian Representation with parameters (μ, u).

[19]There are many different presentations of Blackwell's characterization. See de Oliveira (2018), Bielinska-Kwapisz (2003), Crémer (1982), or Leshno and Spector (1992) for accessible treatments.

(a) (b)

Figure 5: More aligned utilities. In (a), v and u are nearly opposite preferences, as indicated by the narrow agreement region. In (b), the agreement region expands.

Definition 6. Utility indices v and u *agree on the ranking* of lotteries $p, q \in \Delta X$ if either $[v(p) > v(q)$ and $u(p) > u(q)]$ or $[v(q) > v(p)$ and $u(q) > u(p)]$.

Definition 6 states that Sender and Receiver agree on the ranking of p and q if u and v exhibit the same strict ranking of p and q: either both functions rank p strictly better than q or both rank q strictly better than p.

Definition 7. Suppose $(\succsim, c)$ and $(\dot{\succsim}, \dot{c})$ are representable by (ν, μ, v, u) and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$. The preferences of $(\dot{\succsim}, \dot{c})$ are *more aligned* than those of $(\succsim, c)$ if, for all $p, q \in \Delta X$, $\dot{v}$ and $\dot{u}$ agree on the ranking of p and q if v and u do.

Figure 5 illustrates Definition 7. Note that this definition does not require v and $\dot{v}$ (or u and $\dot{u}$) to rank p and q the same way. For example, v and u may rank p over q while $\dot{v}$ and $\dot{u}$ rank q over p. All that matters is that within each Sender-Receiver pair there is no disagreement regarding the ranking of p and q. It is easy to see that if $\dot{v}$ and $\dot{u}$ are "between" the indices v and u in that $\dot{v} = \alpha v + (1-\alpha)u$ and $\dot{u} = \beta v + (1-\beta)u$ for some $\alpha, \beta \in [0,1]$, then $\dot{v}$ and $\dot{u}$ are more consistent than v and u. Hence, the definition is not vacuous.

A bet A is *generic* if $\mathcal{E}^c(A)$ is nonempty. It is straightforward to show that a pq-bet is generic if and only if Receiver is not indifferent between p and q.

Proposition 1. *If $(\succsim, c)$ and $(\dot{\succsim}, \dot{c})$ are representable by (ν, μ, v, u) and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$, then:*

(i) The preferences of $(\dot{\succsim}, \dot{c})$ are more consistent than those of $(\succsim, c)$ if and only if for all generic bets A and experiments σ, $\sigma^* \mathrel{\dot{\succsim}^A} \sigma$ if $\sigma^* \succsim^A \sigma$.

(ii) $v \approx u$ if and only if for every generic bet A and experiment σ, $\sigma^* \succsim^A \sigma$.

Part (i) of Proposition 1 states that the preferences of Sender and Receiver become more aligned as σ^* becomes more attractive to Sender in generic bets. In fact, for a generic pq-bet A, σ^* is top-ranked by $\succsim^A$ if and only if Sender and Receiver agree on the ranking of p and q. Part (ii) expresses the limit case where $v \approx u$ and, hence, Sender prefers perfect information in all generic bets.

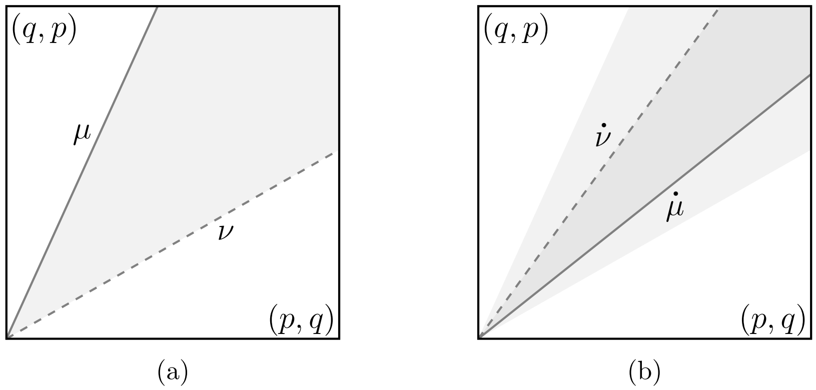


Figure 6: More consistent priors. In (a), there is a relatively large gap between ν and μ—the slopes of the two lines are ν_1/ν_2 and μ_1/μ_2, respectively. In (b), the gap narrows.

Definition 8. Let $E, F \subseteq \Omega$ and $s \in S$. Then ν and μ *agree on the ranking of E and F at s* if either $[\nu^s(E) > \nu^s(F)$ and $\mu^s(E) > \mu^s(F)]$ or $[\nu^s(F) > \nu^s(E)$ and $\mu^s(F) > \mu^s(E)]$, where ν^s and μ^s denote the Bayesian posteriors of ν and μ at signal s.

This definition states that ν and μ agree on the ranking of E and F at s if and only if the Bayesian posteriors ν^s and μ^s rank E and F the same way: either both assign greater probability to E, or both assign greater probability to F.

Definition 9. Suppose $(\succsim, c)$ and $(\dot{\succsim}, \dot{c})$ are representable by (ν, μ, v, u) and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$. The priors of $(\dot{\succsim}, \dot{c})$ are *more aligned* than those of $(\succsim, c)$ if, for all $E, F \subseteq \Omega$ and all $s \in S$, $\dot{\nu}$ and $\dot{\mu}$ agree on the ranking of E and F at s whenever ν and μ do.

In other words, the priors of Sender and Receiver are more aligned if, for any pair of events, there is a larger set of signals making Sender and Receiver agree on the ranking of those events. So, having more aligned priors makes it "easier" for Sender and Receiver to agree on the ranking of any pair of events (see Figure 6). The definition is not vacuous: if, for example, $\alpha, \beta \in [0, 1]$, $\dot{\nu} = \alpha\nu + (1-\alpha)\mu$, and $\dot{\mu} = \beta\nu + (1-\beta)\mu$, then the priors of $(\dot{\succsim}, \dot{c})$ are more aligned than those of $(\succsim, c)$.

Let $\mathcal{E}' \subseteq \mathcal{E}$. Then $\succsim^A$ *satisfies the Blackwell ordering* on $\mathcal{E}'$ if for all $\sigma, \sigma' \in \mathcal{E}'$, $\sigma \sqsupseteq \sigma' \Rightarrow \sigma \succsim^A \sigma'$; $\succsim^A$ *reverses the Blackwell ordering* on $\mathcal{E}'$ if for all $\sigma, \sigma' \in \mathcal{E}'$, $\sigma \sqsupseteq \sigma' \Rightarrow \sigma' \succsim^A \sigma$. A bet A is *non-degenerate* if it is generic and there exists $\sigma, \sigma' \in \mathcal{E}$ such that $\sigma \succ^A \sigma'$. An experiment σ is *EF-informative* if there exists an EF-bet A and $s, t \in \sigma$ such that $c^s(A) \neq c^t(A)$.

Definition 10. An EF-informative experiment σ is *EF-extreme* if there exists a neighborhood $N(\sigma)$ such that, for all non-degenerate EF-bets A, $\succsim^A$ satisfies the Blackwell ordering on $N(\sigma)$ if σ^* is top-ranked by $\succsim^A$, and reverses the Blackwell ordering on $N(\sigma)$ otherwise.

To understand Definition 10, consider an EF-bet. When do Sender and Receiver agree on the ranking of E and F? Intuitively, such agreement occurs at signals that are more informative or "extreme." For example, a signal that perfectly reveals state ω results in the same degenerate posterior for Sender and Receiver. Bayes' rule is continuous in s, so any signal that results in a common strict ranking of E and F can be perturbed without reversing the ranking. So, any disagreement regarding the ranking of E and F must occur at noisier signals, dividing S into "agreement" and "disagreement" regions (see Figure 6). An experiment σ is EF-extreme if every $s \in \sigma$ belongs to the agreement region.

To detect whether σ is EF-extreme, consider a non-degenerate EF-bet $A = \{pEq, pFq\}$. By non-degeneracy, neither Sender nor Receiver is indifferent between p and q. Thus, Sender and Receiver either agree or disagree on the ranking of p and q. If they agree on the ranking, then they agree on the optimal act in A if they agree on which event (E or F) is more likely. Thus, near extreme experiments σ, Receiver's choices from A coincide with what Sender would choose if, hypothetically, he were allowed to choose from A. Consequently, choice behavior coincides with that of a standard Bayesian and $\succsim^A$ satisfies the Blackwell ordering near σ. If instead Sender and Receiver disagree on the ranking of p and q, then Receiver's behavior near σ coincides with that of a standard Bayesian with utility index $-v$ and Sender's preferences reverse the Blackwell ordering near σ.

Proposition 2. *If $(\succsim, c)$ and $(\dot{\succsim}, \dot{c})$ are representable by (ν, μ, v, u) and $(\dot{\nu}, \dot{\mu}, \dot{v}, \dot{u})$, then:*

(i) The priors of $(\dot{\succsim}, \dot{c})$ are more consistent than those of $(\succsim, c)$ if and only if every EF-extreme experiment is also $\dot{E}F$-extreme.

(ii) $\nu = \mu$ if and only if for all generic bets A, either $\succsim^A$ satisfies the Blackwell ordering on $\mathcal{E}$ or reverses it on $\mathcal{E}$.

Part (i) of Proposition 2 captures the idea illustrated by Figure 6 and formalized by the concept of extreme experiments. When more experiments are extreme, more signals generate agreement between Sender and Receiver regarding the ranking of arbitrary events E and F. Hence, Sender's preferences are Blackwell monotone around a larger set of experiments in EF-bets. Part (ii) expresses the limit case where Sender and Receiver share a common prior. With a common prior, every signal realization results in a common posterior. Thus, in any generic EF-bet $A = \{pEq, pFq\}$, Sender's preferences $\succsim^A$ either agree with the Blackwell ordering on all of $\mathcal{E}$ or reverse it on all of $\mathcal{E}$, corresponding to whether they agree or disagree on the ranking of p and q. Note that this logic only applies to bets: if A is not a bet, $\succsim^A$ need not satisfy (or reverse) the Blackwell ordering on $\mathcal{E}$ even with a common prior.[20]

[20]Note also that Sender's preferences in the example from Section 1 (utilizing a common prior) neither

Proposition 3. *If* $(\succsim, c)$ *is representable by* (ν, μ, v, u)*, then* $\nu = \mu$ *and* $v \approx u$ *if and only if either of the following (equivalent) conditions hold:*

(i) For all menus A*,* $\sigma \sqsupseteq \sigma'$ *implies* $\sigma \succsim^A \sigma'$*.*

(ii) For all bets A*,* $\sigma \sqsupseteq \sigma'$ *implies* $\sigma \succsim^A \sigma'$*.*

Proposition 3 provides a characterization of the joint limit case where Sender and Receiver share a common prior and a common utility function. In particular, $\nu = \mu$ and $v \approx u$ if and only if Sender's preferences $\succsim^A$ satisfy the Blackwell ordering in all menus A. Part (ii) establishes that, in fact, adherence to the Blackwell ordering in bets will suffice. Thus, in the behavioral interpretation of the model, the individual is dynamically consistent if and only if each preference $\succsim^A$ satisfies the Blackwell ordering.

6 Conclusion

Leveraging both Sender's preferences for information and Receiver's signal-contingent choices, this paper has characterized the testable implications of a large class of communication models with sender commitment power (Bayesian persuasion). An intermediate result characterizes Receiver as a Bayesian information processor, providing a novel foundation for such behavior. Sender and Receiver can be interpreted as a single individual, reflecting the behavior of a dynamically inconsistent decision maker who—lacking hard commitment power—influences future choice through selective exposure to information.

The results highlight the power and usefulness of information structures (Blackwell experiments) as objects of choice. Although information is of purely instrumental value to standard Bayesian decision makers, Sender's preferences for information fully reveal the priors and utility functions of both agents. Testable conditions on Sender's preferences also characterize the degree of separation between the beliefs or utilities of the two agents.

An advantage of the informational-preference approach is that it characterizes the interaction in terms of the choices agents actually make in disclosure models. Moreover, people frequently compare and choose information structures in daily life. The results of this paper demonstrate how observation of such choices can be used to test models and identify parameters, expanding the types of data that can be used in revealed-preference analysis.

satisfy nor reverse the Blackwell ordering on $\mathcal{E}$. This is because the menu in that example cannot be expressed as a bet.

A Proof of Theorem 1

Preliminaries

This section reviews some basic definitions and results about affine spaces. If $Y \subseteq \mathbb{R}^n$, the *affine hull* of Y is the set $\text{aff}(Y) = \{\alpha^0 x^0 + \ldots + \alpha^m x^m : x^0, \ldots, x^m \in Y \text{ and } \sum_{i=1}^m \alpha^i = 1\}$. Elements of $\text{aff}(X)$ are called *affine combinations* of X. Clearly, $\text{co}(Y) \subseteq \text{aff}(Y)$, where $\text{co}(Y)$ is the convex hull of Y.

A set $Y \subseteq \mathbb{R}^n$ is an *affine space* if $Y = \text{aff}(Y)$. Moreover, every affine space Y is of the form $Y = a + Z := \{a + z : z \in Z\}$ for some $a \in \mathbb{R}^n$ and linear subspace $Z \subseteq \mathbb{R}^n$. Since Z is uniquely determined by Y, we may define the *dimension* of an affine space to be $\dim(Y) := \dim(Z)$, where $Y = a + Z$. We extend this definition to arbitrary convex subsets $C \subseteq \mathbb{R}$ by letting $\dim(C) := \dim(\text{aff}(C))$. That is, the dimension of a convex set is the dimension of its affine hull.

The set ΔX can be identified with a convex subset of $\mathbb{R}^N$ (where $N = |X|$) and satisfies $\dim(\Delta X) = |X| - 1$. Similarly, F can be identified with the set $\Delta X \times \ldots \times \Delta X = \Delta X^{|\Omega|}$ and has dimension $|\Omega|(N-1)$. Finally, a convex subset $C \subseteq (\Delta X)^m$ ($m \geq 1$) has *full dimension* if $\dim(C) = \dim((\Delta X)^m)$; that is, if $(\Delta X)^m \subseteq \text{aff}(C)$.

A set $\{x^0, \ldots, x^m\} \subseteq \mathbb{R}^n$ is *affinely independent* if $\{x^1 - x^0, \ldots, x^m - x^0\}$ is linearly independent. If $Y \subseteq \mathbb{R}^n$ is an affine space of dimension $m - 1$ and $B = \{x^0, \ldots, x^m\} \subseteq Y$ is affinely independent, then B is an *affine basis* for Y. In that case, every $x \in X$ may be expressed in *affine coordinates*: for each $x \in X$, there are unique scalars $\alpha^0, \ldots, \alpha^m \in \mathbb{R}$ with $\sum \alpha^i = 1$ such that $x = \alpha^0 x^0 + \ldots \alpha^m x^m$. Every affine space has an affine basis.

Let $C \subseteq \mathbb{R}^n$ be convex. A function $T : C \to \mathbb{R}$ is *linear* if $T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y)$ whenever $x, y \in C$ and $\alpha \in [0,1]$. A function $T^* : C \to \mathbb{R}$ is *affine* if $T^*(\alpha^0 x^0 + \ldots \alpha^n x^n) = \alpha^0 T^*(x^0) + \ldots + \alpha^n T^*(x^n)$ whenever $x^i \in C$, $\alpha^0 x^0 + \ldots + \alpha^n x^n \in C$, and $\alpha^0 + \ldots + \alpha^n = 1$. Every affine function is linear; the converse also holds.

Step 1: Construction of candidate representation

Recall that $N = |X|$ and let u, μ denote Receiver's (non-constant) utility index and (full support) prior. This step of the proof constructs a menu A^* where the associated set of induced acts is rich enough to pin down candidates for ν and v.

For every $A \in \mathcal{A}$, let $F^A := \{c^\sigma(A) : \sigma \in \mathcal{E}^c(A)\}$; this is the set of induced acts for A. Observe that if $\sigma, \sigma' \in \mathcal{E}^c(A)$ and $\alpha \in (0,1)$, then $c^{\alpha\sigma \cup (1-\alpha)\sigma'}(A) = \alpha c^\sigma(A) + (1-\alpha)c^{\sigma'}(A)$. Thus, F^A is a convex subset of F (in particular, a mixture space under the standard mixture operation on acts). By Consistency, as well as Axioms 1.1–1.3, the restriction of $\succsim^A$ to

$\mathcal{E}^c(A)$ translates into an ordering on F^A satisfying the standard Rationality, Independence, and Continuity axioms. Hence, by the Mixture Space Theorem of Herstein and Milnor (1953), there is a linear function $W^A : F^A \to \mathbb{R}$ such that for all $\sigma, \sigma' \in \mathcal{E}^c(A)$, $\sigma \succsim^A \sigma'$ if and only if $W^A(c^\sigma(A)) \geq W^A(c^{\sigma'}(A))$.

Lemma 1. *There exists an affinely independent set $P = \{p^1, \ldots, p^N\}$ of interior lotteries such that (i) $u(p^N) > u(p^{N-1}) > \ldots > u(p^1)$ and (ii) $u(p^2) - u(p^1) > u(p^3) - u(p^2) > \ldots > u(p^N) - u(p^{N-1})$.*

Proof. It is easy to find interior lotteries satisfying conditions (i) and (ii). If necessary, perturb these lotteries along indifference curves (hyperplanes) in ΔX to arrive at an affinely independent set. Such perturbations are possible because N lotteries in ΔX fail to be affinely independent if and only if they sit on an $(N-2)$-dimensional hyperplane in ΔX. Since indifference curves are linear and the lotteries are interior, one lottery can be perturbed along its indifference plane to yield an affinely independent set. □

For the remainder of Step 1 of the proof, let $P = \{p^1, \ldots, p^N\}$ satisfy the requirements of Lemma 1. The convex hull co(P) has dimension $N-1$ (full dimension in ΔX) because P is affinely independent. It will be useful to think of co(P) as a polytope and each p^i as a vertex of the polytope. Every nonempty $P' \subseteq P$ corresponds to a face of the polytope—in particular, the convex hull co(P') yields a face of dimension $|P'| - 1$.

Lemma 2. *Let $D \subseteq \Delta X$ be a convex subset of co(P') for some $P' \subseteq P$ such that* $\dim D = \dim$ *co*(P'). *If $\hat{p} \in P \backslash P'$ and $q^1, \ldots, q^n \in$ co$(P' \cup \{\hat{p}\}) \backslash$co$(P')$, then* $\dim \bigcap_{i=1}^n \text{co}(D \cup \{q^i\}) = \dim D + 1$.

Proof. First, we prove the following claim: if $x \in D$, $q \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$, and $\varepsilon > 0$, then $\dim(\text{co}(D \cup \{q\}) \cap N^\varepsilon(x)) = \dim D + 1$, where $N^\varepsilon(x)$ is the ε-neighborhood of x.

To prove the claim, note that $\dim(\text{co}(D \cup \{q\})) = \dim D + 1$. Therefore, there exists $z^1, \ldots, z^K \in \text{co}(D \cup \{q\}) \backslash \{x\}$ ($K = \dim D$) such that the set $\{x, z^1, \ldots, z^K\}$ is affinely independent. Thus, the set $\{z^1 - x, \ldots, z^K - x\}$ is linearly independent. Clearly, for every $i = 1, \ldots, K$, the line L^i through x and z^i passes through $N^\varepsilon(x)$. For each i, let $x^i \in N^\varepsilon(x) \backslash \{x\}$ be a point on L^i. Then, since $\{z^1 - x, \ldots, z^K - x\}$ is linearly independent, the set $\{x, x^1, \ldots, x^K\}$ is affinely independent. It follows that $\text{co}\{x, x^1, \ldots, x^K\} \subseteq N^\varepsilon(x)$ has dimension $\dim D + 1$, and therefore $N^\varepsilon(x) \cap \text{co}(D \cup \{q\})$ has dimension $\dim D + 1$. This proves the claim.

Now fix a point x in the (relative) interior of D and apply the claim to each $q = q^1, \ldots, q^W \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$. Since x is in the interior of D, there exists $\varepsilon_i > 0$ $(i = 1, \ldots, n)$ such that $N^{\varepsilon_i}(x) \cap \text{co}(D \cup \{\hat{p}\}) = N^{\varepsilon_i}(x) \cap \text{co}(D \cup \{q^i\})$. Let ε denote the smallest such ε_i and choose a point y in the relative interior of $\text{co}(P' \cup \{\hat{p}\}) \cap N^{\varepsilon}(x)$. By the claim, each set $N^{\varepsilon}(x) \cap \text{co}(D \cup \{q^i\})$ has dimension $\dim D + 1$, and $y \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$. Since $y \in \bigcap_{i=1}^n \text{co}(D \cup \{q^i\})$, it follows that $\dim \bigcap_{i=1}^n \text{co}(D \cup \{q^i\})$ has dimension $D + 1$. $\square$

For an ordered pair $E = [\omega, \omega']$ (where $\omega \neq \omega'$), lotteries p, q, and an act h, let $(p, q)Eh$ denote the act f such that $f_\omega = p$, $f_{\omega'} = q$, and $f_{\hat{\omega}} = h_{\hat{\omega}}$ for all $\hat{\omega} \neq \omega, \omega'$. Similar notation applies for signals: if $\alpha, \beta \in [0, 1]$ and $t \in S$, then $(\alpha, \beta)Et$ denotes the profile r where $r_\omega = \alpha$, $r_{\omega'} = \beta$, and $r_{\hat{\omega}} = t_{\hat{\omega}}$ for all $\hat{\omega} \notin E$. To qualify as a signal, at least one entry of r must be nonzero.

Definition 11 (Symmetric Menu)**.** Let $u(p) > u(\underline{p})$ for all $p \in P$. For each $E = [\omega, \omega']$, let $A^E := \{(p^i, p^{N-i+1})E\underline{p} : i = 1, \ldots, N\} = \{(p^1, p^N)E\underline{p}, (p^2, p^{N-1})E\underline{p}, \ldots, (p^N, p^1)E\underline{p}\}$. The *symmetric menu on* $(P, \underline{p})$ is given by $A^* := \bigcup_E A^E$.

For the remainder of the proof, we take as given a menu A^* satisfying the requirements of Definition 11.

Definition 12. For $E = [\omega, \omega']$, where $\omega \neq \omega'$, let $S^E := \{s \in S : \hat{\omega} \notin E \Rightarrow s_{\hat{\omega}} = 0\}$ and $\mathcal{E}^E := \{\sigma \in \mathcal{E} : \forall s \in \sigma, \text{ either } s \in S^E \text{ or } s = \lambda e^{\hat{\omega}} \text{ for some } \lambda \in (0, 1] \text{ and } \hat{\omega} \in \Omega\}$.

Definition 12 says that if $s \in S^E$, then states outside of E are assigned likelihood 0 by s. An experiment $\sigma \in \mathcal{E}^E$ is composed of signals from S^E as well as (scalar multiples of) indicator signals $e^{\hat{\omega}}$ for each $\hat{\omega} \notin E$, where $e^{\hat{\omega}}$ is a signal assigning likelihood 1 to state $\hat{\omega}$ and 0 to all other states. Observe that S^E and $\mathcal{E}^E$ are convex. It is also easy to verify that if $s \in S^E$, then $c^s(A^*) \subseteq A^E$.

Lemma 3. *For each $E = [\omega, \omega']$ and $f \in A^E$, there is an $s \in S^E$ such that $c^s(A^*) = f$.*

Proof. As noted above, we have $c^s(A^*) \subseteq A^E$ whenever $s \in S^E$. Therefore, we only need to show that for each $f \in A^E$, there is a signal $s \in S^E$ such that $f \succsim^s g$ for all $g \in A^E$.

First, we prove that if $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p}$ for some $s \in S^E$, then $(p^{i+1}, p^{N-(i+1)+1})E\underline{p} \succ^s (p^{i+2}, p^{N-(i+2)+1})E\underline{p}$. So, suppose $s \in S^E$ and $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p}$. Then $s_{\omega'}\mu_{\omega'}[u(p^{N-i+1}) - u(p^{N-(i+1)+1})] > s_\omega \mu_\omega [u(p^{i+1}) - u(p^i)]$. By our choice of P,

$$u(p^{i+1}) - u(p^i) > u(p^{i+2}) - u(p^{i+1}) \tag{7}$$

and

$$u(p^{N-(i+1)+1}) - u(p^{N-(i+2)+1}) > u(p^{N-i+1}) - u(p^{N-(i+1)+1}). \tag{8}$$

Thus,

$$s_{\omega'}\mu_{\omega'}[u(p^{N-(i+1)+1}) - u(p^{N-(i+2)+1})] > s_{\omega'}\mu_{\omega'}[u(p^{N-i+1}) - u(p^{N-(i+1)+1})]$$

$$> s_\omega \mu_\omega [u(p^{i+1}) - u(p^i)]$$
$$> s_\omega \mu_\omega [u(p^{i+2}) - u(p^{i+1})]$$

so that $(p^{i+1}, p^{N-(i+1)+1})E\underline{p} \succ^s (p^{i+2}, p^{N-(i+2)+1})E\underline{p}$.

A similar argument establishes that if $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i-1}, p^{N-(i-1)+1})E\underline{p}$ for some $s \in S^E$, then $(p^{i-1}, p^{N-(i-1)+1})E\underline{p} \succ^s (p^{i-2}, p^{N-(i-2)+1})E\underline{p}$. Thus, for $1 < i < N$, we have $c^s(A^*) = (p^i, p^{N-i+1})E\underline{p}$ if and only if

$$(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p} \quad \text{and} \quad (p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i-1}, p^{N-(i-1)+1})E\underline{p}.$$

Since $s \in S^E$, it cannot be the case that both $s_\omega = 0$ and $s_{\omega'} = 0$. Suppose $s_{\omega'} > 0$. By the Bayesian Representation for c, the above conditions are equivalent to

$$\frac{\mu_{\omega'}}{\mu_\omega} \frac{u(p^{N-(i-1)+1}) - u(p^{N-i+1})}{u(p^i) - u(p^{i-1})} < \frac{s_\omega}{s_{\omega'}} < \frac{\mu_{\omega'}}{\mu_\omega} \frac{u(p^{N-i+1}) - u(p^{N-(i+1)+1})}{u(p^{i+1}) - u(p^i)}.$$

By (7) and (8) (with $i-1$ in place of i), this yields an interval of values for $\frac{s_\omega}{s_{\omega'}}$ such that $c^s(A^*) = (p^i, p^{N-i+1})E\underline{p}$.

For $i = 1$ or $i = N$, observe that $s \in S^E$ satisfies $c^s(A^*) = (p^1, p^N)E\underline{p}$ if and only if $(p^1, p^N)E\underline{p} \succ^s (p^2, p^{N-1})E\underline{p}$ while $c^s(A^*) = (p^N, p^1)E\underline{p}$ if and only if $(p^N, p^1)E\underline{p} \succ^s (p^{N-1}, p^2)E\underline{p}$ (this follows from the first two claims established in this proof). Using the Bayesian Representation in a similar manner, it follows that there exist signals $s \in S^E$ such that $c^s(A^*) = (p^1, p^N)E\underline{p}$. □

Lemma 4. *There exists a convex, full-dimensional set $D \subseteq co(P)$ such that, for every $p \in D$ and every state ω, there is an experiment σ such that $c^\sigma_\omega(A^*) = p$.*

Proof. We will construct D in several steps. First, enumerate $\Omega = \{1, \ldots, W\}$. We will work with pairs of the form $E = [1, \omega]$ for $\omega = 2, \ldots, W$.

Consider $E = [1, \omega]$. Under perfect information σ^*, we have $c^{\sigma^*}_{\hat\omega}(A^*) = p^N$ for all $\hat\omega$. Notice that $\sigma^* \in \mathcal{E}^E$. There exists $\delta > 0$ such that for $s = (1-\delta, \delta)E0$ and $t = (\delta, 1-\delta)E0$, we have $c^s(A^*) = (p^N, p^1)E\underline{p}$ and $c^t(A^*) = (p^1, p^N)E\underline{p}$. Thus, the experiment $\sigma = [s, t] \cup [e_{\hat\omega} : \hat\omega \notin E]$ yields $c^\sigma(A^*) = (\delta p^1 + (1-\delta)p^N, \delta p^1 + (1-\delta)p^N)Ep^N$. Thus, both $c^\sigma_1(A^*)$ and $c^\sigma_\omega(A^*)$ are $\delta p^1 + (1-\delta)p^N$. Mixing σ^* with σ yields a convex set $D^1_\omega \subseteq \text{co}(\{p^1, p^N\})$ of dimension 1 such that, for all $p \in D^1_\omega$, there exists σ such that $c^\sigma_{\omega'}(A^*) = p$ for $\omega' = 1, \omega$. Since every such set lies on the face $\text{co}(\{p^1, p^N\})$ and contains p^N, the set $D^1 := \bigcap_{\omega \geq 2} D^1_\omega$ is nonempty and has dimension 1.

For each $E = [1, \omega]$, pick a signal $s \in S^E$ such that $c^s(A^*) = [p^{N-1}, p^2]E\underline{p}$ (Lemma 3) and an experiment $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$ for some p in the interior of D^1. Thus,

σ contains a signal t such that $c^t(A^*)$ is a singleton and $t_{\hat{\omega}} > 0$ for $\hat{\omega} = 1, \omega$ (otherwise, p is not in the interior of D^1). Therefore, we may assume $t - s$ is a well-defined signal and that $c^{t-s}(A^*) = c^t(A^*)$ (if necessary, replace s with λs for a sufficiently small $\lambda > 0$). Let σ' denote the experiment formed by taking σ, appending s, and replacing t with $t - s$. Then $q^\omega := c^{\sigma'}_\omega(A^*) \in \text{co}\{p^1, p^2, p^N\} \backslash \text{co}\{p^1, p^N\}$. Taking mixtures of σ' and σ (and letting σ vary in order to generate $c^\sigma_\omega(A^*) = p$ for all $p \in \text{int}D^1$) implies that every $q \in \text{co}\{\text{int}D^1, q^\omega\}$ satisfies $c^{\sigma''}_\omega(A^*)$ for some $\sigma'' \in S^E$. Repeating this procedure for every choice of $E = [1, \omega]$ (and also for $E = [\omega, 1]$ for some ω) yields lotteries $q^1, \ldots, q^W \in \text{co}\{p^1, p^2, p^N\} \backslash \text{co}\{p^1, p^N\}$. By Lemma 2, $D^2 := \bigcap_{\omega=1}^W \text{co}(\text{int}D^1 \cup \{q^\omega\})$ has dimension 2. By construction, for every $p \in D^2$ and every ω there is an experiment $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$.

We now proceed by induction. Suppose $D^i \subseteq \text{co}\{p^1, \ldots, p^i, p^N\}$ $(2 \leq i < N)$ is a convex set of dimension i and, for all $p \in D^i$ and all ω, there exists $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$. We construct a convex set $D^{i+1} \subseteq \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\}$ of dimension $i + 1$ such that, for every $p \in D^{i+1}$ and ω, there exists $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$. The procedure is similar to the previous step. First, take $E = [1, \omega]$ and an experiment $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$ for some p in the interior of D^i. Then σ contains a signal t such that $c^t(A^*)$ is a singleton and $t_{\hat{\omega}} > 0$ for $\hat{\omega} \in E$. Pick a signal $s \in S^E$ such that $c^s(A^*) = (p^{N-i}, p^{i+1})E\underline{p}$. We may assume that $t - s$ is a well-defined signal such that $c^{t-s}(A^*) = c^t(A^*)$ (if necessary, scale s down by a factor $\lambda > 0$). Let σ' be an experiment formed by deleting t from σ and appending $t - s$ and s. Then $q^\omega := c^{\sigma'}_\omega(A^*) \in \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\} \backslash \text{co}\{p^1, \ldots, p^i, p^N\}$. Repeating this for all ω (as well as $E = [1, \omega]$ for some ω) yields lotteries $q^1, \ldots, q^W \in \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\} \backslash \text{co}\{p^1, \ldots, p^i, p^N\}$. By Lemma 2, $D^{i+1} := \bigcap_{\omega=1}^W \text{co}(\text{int}D^i \cup \{q^\omega\})$ has dimension $i + 1$. By construction, for every $p \in D^{i+1}$ and every ω there is an experiment $\sigma \in \mathcal{E}^E$ such that $c^\sigma_\omega(A^*) = p$. $\square$

For the remainder of Step 1, let $D \subseteq \text{co}(P)$ satisfy all requirements of Lemma 4.

Definition 13 (Interior Experiment)**.** Fix a menu A. For each $f \in A$, let $S^A(f) := \{s \in S : c^s(A) = f\}$. An experiment σ is A-*interior* if (i) $c^s(A)$ is single-valued for all $s \in \sigma$, and (ii) for each $f \in A$, there is exactly one $s \in \sigma$ such that $c^s(A) = f$. Similarly, any set σ of signals (not necessarily qualifying as an experiment) is A-interior if it satisfies conditions (i) and (ii). Such a set is necessarily nonempty and finite.

Let S^* denote the set of all signals s such that $s_\omega > 0$ for all ω. The statement $\sigma \subseteq S^*$ means σ is a matrix where each column is a member of S^*. Such matrices do not necessarily qualify as experiments.

Definition 14 (ε-Neighborhood)**.** Suppose $\sigma \subseteq S^*$ is A-interior and let $\varepsilon > 0$. For each $s \in \sigma$, let $Q^{s,\varepsilon} := \prod_\omega (s_\omega - \varepsilon, s_\omega + \varepsilon)$. Let B^ε denote the set of all A-interior matrices $\sigma' \subseteq S^*$

such that (i) for each ω, $\sum_{s'\in\sigma'} s'_\omega = \sum_{s\in\sigma} s_\omega$, and (ii) if $s \in \sigma$, $s' \in \sigma'$, and $c^s(A) = c^{s'}(A)$, then $s' \in Q^{s,\varepsilon}$. Then B^ε is an *ε-neighborhood* of σ in A.

Note that Definition 14 does not require σ to be an experiment, and that $B^\varepsilon \subseteq \mathcal{E}$ (in fact, $B^\varepsilon \subseteq \mathcal{E}^c(A)$) if and only if σ is an experiment. The next two lemmas provide general results about menus and neighborhoods of experiments inducing full-dimensional sets of acts. These are also used in Step 2 of the proof.

Lemma 5. *Suppose that, for each ω, $L^*_\omega \subseteq \Delta X$ is full-dimensional. Let $f^* \in F$ and define $L^*_\omega[\omega]f^* := \{p[\omega]f^* : p \in L^*_\omega\}$. If $G \subseteq F$ is convex and $L^*_\omega[\omega]f^* \subseteq G$ for all ω, then G has full dimension.*

Proof. We need to show that every Anscombe-Aumann act is in the affine hull of G. To begin, note that for each ω, aff(G) contains aff$(L^*_\omega[\omega]f^*) = \{p[\omega]f^* : p \in \Delta X\}$ since L^*_ω has full dimension in ΔX. Therefore aff(G) contains aff(C), where $C = \bigcup_\omega \{p[\omega]f^* : p \in \Delta X\}$. So, it is enough to find a finite set $B \subseteq C$ such that $F \subseteq$ aff(B). A natural candidate for B involves the (affinely independent) set $P = \{p^1, \ldots, p^N\} \subseteq \Delta X$. In particular, let $B = \bigcup_{\omega\in\Omega} \{p^i[\omega]f^* : i = 1, \ldots, N\}$. Clearly, $B \subseteq C$.

To see that $F \subseteq$ aff(B), let $f \in F$ and $\alpha = (\alpha_\omega)_{\omega\in\Omega} \in (0,1)^\Omega$ such that $\sum_\omega \alpha_\omega = 1$. For each ω, we have $\frac{\sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega}{1-\alpha_\omega} = f^*_\omega \in \Delta X$. Therefore, there is some $\hat{f}_\omega \in \Delta X$ such that $f_\omega = \alpha_\omega \hat{f}_\omega + (1-\alpha_\omega)\frac{\sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega}{1-\alpha_\omega} = \alpha\hat{f}_\omega + \sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega$. Since P is affinely independent with dim(aff(P)) = dim(ΔX), for each ω there are numbers β^i_ω ($i = 1, \ldots, N$) such that $\hat{f}_\omega = \sum_{i=1}^N \beta^i_\omega p^i$ and $\sum_{i=1}^N \beta^i_\omega = 1$. Thus,

$$f_\omega = \alpha_\omega \hat{f}_\omega + \sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega = \alpha_\omega \sum_{i=1}^N \beta^i_\omega p^i + \sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega = \sum_{i=1}^N \alpha^i_\omega p^i + \sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega,$$

where $\alpha^i_\omega := \alpha_\omega\beta^i_\omega$. Note that $\sum_{i=1}^N \alpha^i_\omega = \alpha_\omega$ for each ω, so that $\sum_\omega \sum_{i=1}^N \alpha^i_\omega = \sum_\omega \alpha_\omega = 1$. Then

$$\sum_\omega \sum_{i=1}^N \alpha^i_\omega p^i[\omega]f^* = \left(\sum_{i=1}^N \alpha^i_\omega p^i + \sum_{\omega'\neq\omega}\sum_{i=1}^N \alpha^i_{\omega'} f^*_\omega\right)_{\omega\in\Omega} = \left(\alpha_\omega\hat{f}_\omega + \sum_{\omega'\neq\omega}\alpha_{\omega'}f^*_\omega\right)_{\omega\in\Omega} = f.$$

Thus, $f \in$ aff(B). □

Lemma 6. *Suppose $\sigma \in \mathcal{E}$ is A-interior and that, for each ω, there is a nonempty $B \subseteq A$ such that $|B| = N$ and $B_\omega := \{f_\omega : f \in B\}$ is affinely independent. If B^ε is an ε-neighborhood for σ, then:*

(i) For each ω, $F^A(B^\varepsilon) := \{c^{\sigma'}(A) : \sigma' \in B^\varepsilon\}$ has a subset of the form $\{p[\omega]f^* : p \in L^*\}$, where $L^* \subseteq \Delta X$ is full-dimensional and $f^* = c^\sigma(A)$; and

(ii) $F^A(B^\varepsilon)$ contains a full-dimensional ball around $c^\sigma(A)$.

Proof.

(i) Fix a state ω and let $f^* = c^\sigma(A)$ and $f^*_{-B} := \sum_{s\in\sigma^{-B}} s_\omega c^s_\omega(A)$, where $\sigma^B := \{s \in \sigma : c^s(A) \in B\}$ and $\sigma^{-B} := \sigma\backslash\sigma^B$. Then $|\sigma^B| = N$. Without loss of generality, let B^ε denote an ε-neighborhood of σ^B. Let $\omega \in \Omega$ and note that for every $\sigma' \in B^\varepsilon$, there is a natural bijection between signals of σ and signals of σ'; specifically, $s \in \sigma$ and $s' \in \sigma'$ are related if and only if $c^s(A) = f^s = c^{s'}(A)$. For each $s \in \sigma$, let s' denote the corresponding signal in σ'.

Consider $\sigma' \in B^\varepsilon$ such that for all $s \in \sigma$ and all $\omega' \neq \omega$, $s_{\omega'} = s'_{\omega'}$. Thus, every such σ' induces an act of the form $p[\omega]f^*$, where

$$
\begin{aligned}
p &\in \left\{\sum_{s'\in\sigma'} s'_\omega f^s_\omega + f^*_{-B} : s'_\omega \in (s_\omega - \varepsilon, s_\omega + \varepsilon) \text{ for all } s' \in \sigma', \text{ and } \sum_{s'\in\sigma'} s'_\omega = \sum_{s\in\sigma^B} s_\omega\right\} \\
&= \left\{\sum_{s\in\sigma^B} (s_\omega + \delta^s) f^s_\omega + f^*_{-B} : |\delta^s| < \varepsilon \text{ and } \sum_{s\in\sigma^B} \delta^s = 0\right\} \\
&= \left\{f^*_\omega + \sum_{s\in\sigma^B} \delta^s f^s_\omega : |\delta^s| < \varepsilon \text{ and } \sum_{s\in\sigma^B} \delta^s = 0\right\}.
\end{aligned}
$$

So, it will suffice to show that the set $C := \left\{\sum_{s\in\sigma^B} \delta^s f^s_\omega : |\delta^s| < \varepsilon \text{ and } \sum_{s\in\sigma^B} \delta^s = 0\right\}$ has dimension $N-1$ (clearly, C is convex). Note that $N-1$ is an upper bound on the dimension of C because C is a translation of a subset of ΔX. Pick any $s^* \in \sigma^B$ and note that if $\sum_{s\in\sigma^B} \delta^s = 0$, then $\delta^{s^*} = -\sum_{s\in\sigma^B\backslash s^*} \delta^s$. Thus

$$
\begin{aligned}
C &= \left\{\sum_{s\in\sigma^B\backslash s^*} \delta^s f^s_\omega - \sum_{s\in\sigma^B\backslash s^*} \delta^s f^{s^*}_\omega : |\delta^s| < \varepsilon \ \forall s \neq s^*, \text{ and } \left|\sum_{s\in\sigma^B\backslash s^*} \delta^s\right| < \varepsilon\right\} \\
&= \left\{\sum_{s\in\sigma^B\backslash s^*} \delta^s (f^s_\omega - f^{s^*}_\omega) : |\delta^s| < \varepsilon \ \forall s \neq s^*, \text{ and } \left|\sum_{s\in\sigma^B\backslash s^*} \delta^s\right| < \varepsilon\right\}.
\end{aligned}
$$

Let $\lambda^s := f^s_\omega - f^{s^*}_\omega$ for each $s \in \sigma^B\backslash s^*$. Then $\{\lambda^s : s \in \sigma^B\backslash s^*\}$ is linearly independent because $B_\omega = \{f^s_\omega : s \in \sigma^B\}$ is affinely independent. Let $C' := \{0\} \cup \left\{\frac{\varepsilon/2}{N-1}\lambda^s : s \in \sigma^B\backslash s^*\right\}$. Then C' is an affinely independent set of N vectors in $\mathbb{R}^N$, so that its convex hull has dimension $N-1$. Moreover, C contains the convex hull of C' because if $\lambda \in$

co(C'), then there are scalars $\alpha^s \in [0,1]$ $(s \in \sigma^B)$ such that $\sum_{s\in\sigma^B} \alpha^s = 1$ and $\lambda = \alpha^{s^*}0 + \sum_{s\in\sigma\setminus s^*} \alpha^s \frac{\varepsilon/2}{N-1}\lambda^s$. We have $\lambda \in C$ because $\left|\frac{\alpha^s(\varepsilon/2)}{N-1}\right| < \varepsilon$ for all $s \in \sigma^B \setminus s^*$ and $\left|\sum_{s\in\sigma^B\setminus s^*} \frac{\alpha^s(\varepsilon/2)}{N-1}\right| < \varepsilon/2$. Since C contains the convex hull of C', and C' has dimension $N-1$, it follows that C has dimension $N-1$ (recall that $N-1$ is an upper bound on the dimension of C).

(ii) By part (i), $F^A(B^\varepsilon)$ (hence F^A) contains a subset of the form $L^*_\omega[\omega]f^*$ for each ω, where each set $L^*_\omega \subseteq \Delta X$ has full dimension. Now apply Lemma 5. □

Lemma 7. *There is a full-dimensional set $L^* \subseteq \Delta X$ such that, for all ω, there exists $h^\omega \in F$ such that $L^*[\omega]h^\omega := \{p[\omega]h^\omega : p \in L^*\} \subseteq F^{A^*}$.*

Proof. Choose a lottery p^* in the interior of D (recall that D satisfies all requirements of Lemma 4). Fix a state ω. Then there is an A^*-interior experiment σ such that $c^\sigma(A^*) = p^*[\omega]h$ for some $h \in F$. By part (ii) of Lemma 6, F^{A^*} contains a full-dimensional ball around $p^*[\omega]h$. In particular, there is a convex, full-dimensional set $L^*_\omega \subseteq \Delta X$ such that p^* belongs to the interior of L^*_ω and $\{p[\omega]h : p \in L^*_\omega\} \subseteq F^{A^*}$. We may assume that $L^*_\omega \subseteq D$. Since $p^* \in D$, we can repeat this argument for all ω to get a family of convex, full-dimensional sets $L^*_\omega \subseteq \Delta X$, each containing p^* as an interior point, and acts $h^\omega \in F$ such that $\{p[\omega]h^\omega : p \in L^*_\omega\} \subseteq F^{A^*}$. Letting $L^* := \bigcap_{\omega\in\Omega} L^*_\omega$ completes the proof. □

Lemma 8. *Any linear representation $W^{A^*} : F^{A^*} \to \mathbb{R}$ of $\succsim^{A^*}$ on $\mathcal{E}^c(A^*)$ has a unique linear extension $W : F \to \mathbb{R}$. The extension represents a preference $\succsim$ on F satisfying all of the Anscombe-Aumann axioms except (possibly) the Non-Degeneracy axiom.*

Proof. As explained at the start of Step 1, a linear representation W^{A^*} exists. By Lemmas 5 and 7, F^{A^*} has full dimension, and therefore W^{A^*} has a unique linear extension $W : F \to \mathbb{R}$. This induces a complete and transitive relation $\succsim$ on F by letting $f \succsim g$ if and only if $W(f) \geq W(g)$. The Independence and Continuity axioms are satisfied by linearity of W.

To verify that $\succsim$ satisfies the State Independence axiom, suppose $p[\omega]h \succsim q[\omega]h$ and let $\omega' \in \Omega$ and $h' \in F$. We want to show that $p[\omega']h' \succsim q[\omega']h'$. By a standard result, there exist linear functions $U_\omega : \Delta X \to \mathbb{R}$ (unique up to positive affine transformation) such that $W(f) = \sum_\omega U_\omega(f_\omega)$ for all $f \in F$. Thus, $p[\omega]h \succsim q[\omega]h$ implies $U_\omega(p) \geq U_\omega(q)$.

Since $L^*[\omega]h^\omega \subseteq F^{A^*}$ for each ω, where $L^* \subseteq \Delta X$ is convex and has full dimension, there exists $r \in L^*$ and $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha)r \in L^*$ and $\alpha q + (1-\alpha)r \in L^*$. Thus, $(\alpha p + (1-\alpha)r)[\omega]h^\omega$, $(\alpha q + (1-\alpha)r)[\omega]h^\omega$, $(\alpha p + (1-\alpha)r)[\omega']h^{\omega'}$, and $(\alpha q + (1-\alpha)r)[\omega']h^{\omega'}$ are elements of F^{A^*}. Moreover, $(\alpha p + (1-\alpha)r)[\omega]h^\omega \succsim (\alpha q + (1-\alpha)r)[\omega]h^\omega$ because $W((\alpha p + (1-\alpha)r)[\omega]h^\omega) \geq W((\alpha q + (1-\alpha)r)[\omega]h^\omega)$ if and only if $U_\omega(p) \geq U_\omega(q)$.

Since $\succsim^{A^*}$ satisfies State Independence (Axiom 1.6) on the domain $\mathcal{E}^c(A^*)$, it follows that $(\alpha p + (1-\alpha)r)[\omega']h^\omega \succsim (\alpha q + (1-\alpha)r)[\omega']h^\omega$. Therefore $U_{\omega'}(p) \geq U_{\omega'}(q)$, so that $U_{\omega'}(p) + \sum_{\hat{\omega}\neq\omega} U_{\hat{\omega}}(h'_{\hat{\omega}}) \geq U_{\omega'}(q) + \sum_{\hat{\omega}\neq\omega} U_{\hat{\omega}}(h'_{\hat{\omega}})$. Thus, $p[\omega']h' \succsim q[\omega']h'$, as desired. □

Step 2: Spreading the representation

For the remainder of the proof, assume u has been normalized to take values in $[0,1]$. A binary relation $\succsim$ on F is a *linear preference relation* if it has a linear representation.

Definition 15. Let A and B be menus such that $\mathcal{E}^c(A)$ and $\mathcal{E}^c(B)$ are nonempty.

(i) A relation $\succsim$ on F *agrees with* $\succsim^A$ if, for all $\sigma, \sigma' \in \mathcal{E}^c(A)$, $\sigma \succsim^A \sigma' \Leftrightarrow c^\sigma(A) \succsim c^{\sigma'}(A)$.

(ii) A *inherits* a representation from B if every linear preference relation $\succsim$ on F that agrees with B also agrees with A.

(iii) A and B *share a representation* if there is a unique linear preference relation $\succsim$ on F that agrees with both $\succsim^A$ and $\succsim^B$.

Lemma 9. *Let A and B be menus such that $\mathcal{E}^c(A)$ and $\mathcal{E}^c(B)$ are nonempty.*

(i) If $\dim(F^A) = \dim(F^A \cap F^B) \leq \dim(F^B)$, then A inherits a representation from B.

(ii) If $\dim(F^A) = \dim(F^A \cap F^B) = \dim(F^B) = \dim(F)$, then A and B share a representation.

Proof. By the Consistency axiom, $\succsim^A$ and $\succsim^B$ agree on the domain $F^A \cap F^B$. The restriction of W^B to $F^A \cap F^B$ is a linear function L. Since $F^A \cap F^B$ is convex and $\dim(F^A) = \dim(F^A \cap F^B) \leq \dim(F^B)$, L has a linear extension to F^A. Every such extension represents a linear preference relation $\succsim$ on F^A that agrees with A and B, proving (i). For (ii), note that L has a unique linear extension to F whenever $\dim(F^A \cap F^B) = \dim(F)$. □

Definition 16. Let A be a menu.

1. If $f \in A$, the *support* of f is the set $S^A(f) := \{s \in S : c^s(A) = f\}$.

2. A is a *k-menu* if $|A| = k \geq 2$ and each $f \in A$ has nonempty support.

3. A is *independent* if it is a k-menu for some k and, for each ω, there is an N-menu $B \subseteq A$ such that $B_\omega := \{f_\omega : f \in B\}$ is affinely independent.

Lemma 10. *Suppose A is a k-menu.*

(i) If $f \in A$, then $S^A(f)$ is a convex cone and has full dimension (in S).

(ii) There exists an A-interior experiment σ.

(iii) If A is independent, then F^A has full dimension (in F).

Proof.

(i) Observe that $s \in S^A(f)$ if and only if, for all $g \in A$, $\sum_\omega s_\omega \mu_\omega u(f_\omega) > \sum_\omega s_\omega \mu_\omega u(g_\omega)$. It is straightforward to verify that if $s, t \in S^A(f)$, then $\lambda s \in S^A(f)$ for all $\lambda > 0$ such that $\lambda s \in S$, and $\alpha s + (1-\alpha)t \in S^A(f)$ for all $\alpha \in [0,1]$. Thus, $S^A(f)$ is a convex cone. To see that it is a full-dimensional subset of $S := [0,1]^\Omega \backslash 0$, note that since the above inequalities are strict, there is an open ball around each $s \in S^A(f)$ that preserves the inequality; since the open ball has full dimension, the result follows.

(ii) Since A is finite and each set $S^A(f)$ is a convex cone, there are signals s^f $(f \in A)$ such that $c^{s^f}(A) = f$ and, for each ω, $\sum_{f \in A} s^f_\omega \leq 1$ (simply choose any signals $s^f \in S^A(f)$ and, if necessary, scale them all down by a factor $\alpha \in (0,1)$ to ensure $\sum_{f \in A} s^f_\omega \leq 1$). For each ω, there is an $f \in A$ such that $u(f_\omega) \geq u(g_\omega)$ for all $g \in A$. Thus, s^f_ω can be increased as needed to ensure $\sum_{f \in A} s^f_\omega = 1$. Repeat this for each ω to get a well-defined experiment $\sigma = \{s^f : f \in A\}$.

(iii) By part (ii), there is an A-interior σ and, hence, a ε-neighborhood around σ. Let $\omega \in \Omega$. Since A is independent, there is an N-menu $B \subseteq A$ such that $B_\omega = \{f_\omega : f \in B\}$ is affinely independent. Now apply Lemma 6. $\square$

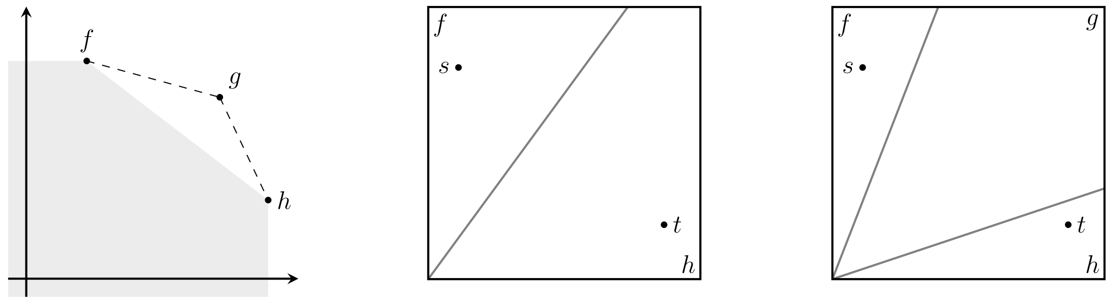
Definition 17. A finite, nonempty set $\mathcal{C}$ of convex cones in S is a *conic decomposition* if $\mathcal{C} = \{S^A(f) : f \in A\}$ for some k-menu A. For each k-menu A, the set $\mathcal{C}(A) := \{S^A(f) : f \in A\}$ is the *conic decomposition for A.*

Definition 18. For each k-menu A and $f \in A$, let $U(f) := (\mu_\omega u(f_\omega))_{\omega \in \Omega}$ denote the *(virtual) utility coordinate* for f, and let $U(A) := \{U(f) : f \in A\}$ denote the *utility profile for A*. If a set $U \subseteq \mathbb{R}^\Omega_+$ satisfies $U = U(A)$ for some k-menu A, then U is a *k-utility profile*. Finally, a finite set $U \subseteq \mathbb{R}^\Omega_+$ is a *utility profile* if U is a k-utility profile for some k.

Lemma 11. *If A and B are k-menus such that $U(A) = U(B)$, then $\mathcal{C}(A) = \mathcal{C}(B)$.*

Proof. This follows immediately from the definition of $U(A)$ and the fact that $s \in S^A(f)$ if and only if $\sum_\omega s_\omega \mu_\omega u(f_\omega) > \sum_\omega s_\omega \mu_\omega u(g_\omega)$ for all $g \in A\backslash\{f\}$. $\square$

By Lemma 11, each utility profile U has an associated conic decomposition $\mathcal{C}(U)$. Specifically, $\mathcal{C}(U)$ is the unique $\mathcal{C}$ such that $U(A) = U$ implies $\mathcal{C}(A) = \mathcal{C}$.



(a) Utility Space

(b) Signal space for $A = \{f, h\}$

(c) Signal space for $B = \{f, g, h\}$

Figure 7: Illustration of Lemma 12. The shaded region in (a) is $T(A)$. Experiment $\sigma = [s, t]$ is constructed so that f is chosen at s and h is chosen at t. If the (utility coordinate) of g is close to the face joining f and h, then $c^s(B) = f$ and $c^t(B) = h$ as well, where $B = \{f, g, h\}$. How close g needs to be to the face depends on s and t (the dashed lines in (a) are perpendicular to the gray lines in (c)). Thus, $c^{\sigma'}(A) = c^{\sigma'}(B)$ for all σ' in a neighborhood of σ, so that A inherits a representation from B.

Definition 19. Let U be a utility profile and $z = (z_\omega)_{\omega\in\Omega} \in U$. The *support* of z in U is the set $S^U(z) := \{s \in S : \forall z' \in U,\ \sum_\omega s_\omega z_\omega > \sum_\omega s_\omega z'_\omega\}$.

Definition 20. Let U be a utility profile. For each $z \in U$ and $s \in S^U(z)$, let $H(z, s) := \{\lambda \in \mathbb{R}^\Omega : s \cdot (\lambda - z) \leq 0\}$. The *support polytope of z in U*, denoted $T(z, U)$, is defined as $T(z, U) := \bigcap_{s\in S^U(z)} H(z, s)$. The *polytope of U*, denoted $T(U)$, is given by $T(U) := \bigcap_{z\in U} T(z, U)$. A polytope $T \subseteq \mathbb{R}^\Omega$ is a *decision polytope* if $T = T(U)$ for some utility profile U; it is a *k-polytope* if $T = T(U(A))$ for some k-menu A.

Definition 21. Let T be a decision polytope. For each face F of T, let $\eta^F \in S^\Omega_+ := \{\eta \in \mathbb{R}^\Omega_+ : \|\eta\| = 1\}$ such that η^F is normal to the hyperplane associated with F. Let $\mathcal{N}(T) := \{\eta^F : F \text{ is a face of } T\}$ denote the set of *normals* for T.

Figure 1 in the main text illustrates the relationship between a menu A, its utility profile $U(A)$, and the associated decision polytope and conic decomposition. In the figure, the shaded region is $T(U(A))$. An act is chosen under some signal if and only if $U(f)$ is an extreme point of the polytope. For any such act f, the set of signals s where $c^s(A) = f$ is a cone in S. Faces of the polytope correspond to signals making Receiver indifferent between two or more acts in A. Thus, any s perpendicular to a face of the polytope lies on a hyperplane in signal space separating the cones corresponding to two or more acts.

Lemma 12 (Vertex Expansion)**.** *Let A be a k-menu. There is an act $g \notin A$ such that $B = A \cup \{g\}$ is a $(k + 1)$-menu and A inherits a representation from B.*

Proof. Let $\sigma \in \mathcal{E}$ be A-interior and choose $2\varepsilon > 0$ such that $B^{2\varepsilon}$ is a 2ε-neighborhood of σ. Then B^ε is an ε-neighborhood where, for all $\sigma' \in B^\varepsilon$ and all $s \in \sigma'$, the closure of $Q^{s,\varepsilon}$ is in the interior of $S^A(f)$, where $f = c^s(A)$.

Let $f \in A$. For each $\sigma' \in B^\varepsilon$ and each $s \in \sigma$, consider the half-space $H(f,s) := \{\lambda \in \mathbb{R}^\Omega_+ : s \cdot (\lambda - U(f)) \leq 0\}$. This is the half-space (containing the origin) where the bounding hyperplane has normal s and passes through $U(f)$. Let T^* be (the closure of) the intersection over all $H(f,s)$ where $f \in A$ and $s \in \sigma' \in B^\varepsilon$. Notice that for each f, the set $B^\varepsilon(f) := \{s \in S : c^s(A) = f \text{ and } s \in \sigma' \in B^\varepsilon\}$ is an (open) convex cone in S, and a strict subset of $\text{int}(S^A(f))$ by our choice of ε. Thus, $B^\varepsilon(f)$ and $B^\varepsilon(f')$ are strictly separated whenever $f \neq f'$, and therefore $T(A) \subsetneq T^*$. Pick any point $u^* \in [T^* \backslash T(A)] \cap \mathbb{R}^\Omega_+$ and let $g \in F$ such that $U(g) = u^*$. Then $B = A \cup \{g\}$ is the desired menu.

To see why A and B share a representation, note that (by construction) $c^s(A) = c^s(B)$ for all $s \in \sigma' \in B^\varepsilon$. Hence, $c^{\sigma'}(A) = c^{\sigma'}(B)$ whenever $\sigma' \in B^\varepsilon$. Since $\dim(F^A) = \dim(F^A(B^\varepsilon))$ and $F^A(B^\varepsilon) = F^B(B^\varepsilon) \subseteq F^B$, it follows that $\succsim^A$ inherits a representation from $\succsim^B$. □

Lemma 13. *Let A be a k-menu. There exists an independent menu B such that A inherits a representation from B.*

Proof. Fix an A-interior experiment σ and a neighborhood B^ε of the form used in the proof of Lemma 12. It is easy to see that a similar argument can be used to add N additional vertices to the region $T^* \backslash T(A)$ to yield a $(k+N)$-polytope. Moreover, these vertices can be chosen so that for each state ω, the ω coordinates yield N distinct, interior utility values. We are free to pick any N lotteries $p^1_\omega, \ldots, p^N_\omega$ yielding these utility values. Clearly, these can be chosen to form an affinely independent set. Now let $f^i = (p^i_\omega)_{\omega \in \Omega} \in F$, and let $B = A \cup \{f^1, \ldots, f^N\}$. □

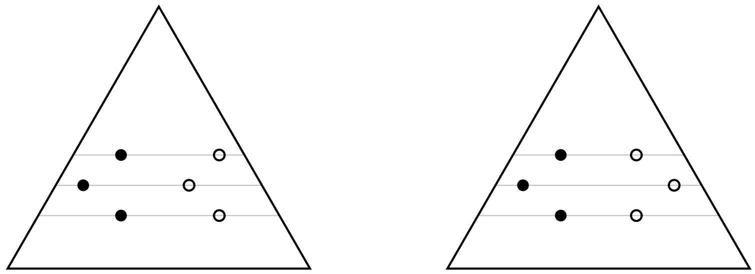
Definition 22. Let A and B be independent menus. Then B is a *translation* of A if there exists $\lambda^* \in \mathbb{R}^\Omega$ such that $T(B) = T(A) + \lambda^* := \{\lambda + \lambda^* : \lambda \in T(A)\}$. The notation $B = A + \lambda^*$ means $T(B) = T(A) + \lambda^*$.

Lemma 14. *If $B = A + \lambda^*$, then:*

(i) The map $\psi : U(A) \to U(B)$ given by $\psi(z) := z + \lambda^*$ is a bijection. Hence, there is a bijection $\psi : A \to B$ where $\psi(f)$ denotes the unique $g \in B$ such that $U(g) = U(f) + \lambda^*$.

(ii) $\mathcal{C}(B) = \mathcal{C}(A)$.

Proof. Part (i) is clear. For part (ii), observe that $s \in S^A(f)$ if and only if $\sum_\omega s_\omega u(f_\omega) > \sum_\omega s_\omega u(g_\omega) \Leftrightarrow \sum_\omega s_\omega[\mu_\omega u(f_\omega) + \lambda^*_\omega] > \sum_\omega s_\omega[\mu_\omega u(g_\omega) + \lambda^*_\omega] \Leftrightarrow s \in S^B(\psi(f))$. It follows that $\mathcal{C}(B) = \mathcal{C}(A)$. □



(a) Oriented (b) Not Oriented

Figure 8: Orientedness when $|X| = 3$. The lotteries $A_\omega = \{f_\omega, g_\omega, h_\omega\}$ are solid dots and the lotteries $B_\omega = \{f'_\omega, g'_\omega, h'_\omega\}$ are circles. In this case, $\lambda = 0$ (the lines are indifference curves for u). The configuration in (b) is not oriented because the affine path from A_ω to B_ω (traversing along the lines) yields a collinear set of lotteries at $\alpha = 1/2$.

Definition 23. Suppose B is a translation of A, and let $\psi : A \to B$ denote the associated bijection (Lemma 14). The *affine path from f to $\psi(f)$* is the map $\alpha \mapsto f^\alpha := (1-\alpha)f + \alpha\psi(f)$ for $\alpha \in [0,1]$ and the *affine path from A to B* is the map $\alpha \mapsto A^\alpha := \{f^\alpha : f \in A\}$ for $\alpha \in [0,1]$.

Definition 24. A bijection $\varphi : P \to Q$ between to sets of N lotteries is *oriented* if (i) for all $p, p' \in P$, $u(p) > u(p')$ implies $u(\varphi(p)) > u(\varphi(p'))$, and (ii) for each $\alpha \in [0,1]$, the set $\{(1-\alpha)p + \alpha\varphi(p) : p \in P\}$ is affinely independent. Independent menus A and B are *oriented* if B is a translation of A and, for each ω, the map $\varphi_\omega : A_\omega \to B_\omega$ given by $\varphi_\omega(f_\omega) := \psi(f)_\omega$ is oriented, where $A_\omega := \{f_\omega : f \in A\}$, $B_\omega := \{g_\omega : g \in B\}$, and $\psi : A \to B$ is the associated bijection (Lemma 14).

Figure 8 illustrates the concept of orientedness. Note that not all translations $B = A + \lambda^*$ are oriented; in fact, it is possible to construct menus A and B such that $U(A) = U(B)$ (so that B is trivially a translation of A) but where A and B are not oriented.

Lemma 15. *If A and B are oriented menus, then A and B share a representation.*

Proof. Since A and B are oriented, there is a $\lambda^* \in \mathbb{R}^\Omega$ such that $B = A + \lambda^*$ and an associated bijection $\psi : A \to B$ (Lemma 14). Consider the affine path associated with ψ (Definition 23), and note that for each α, $A^\alpha = A + \alpha\lambda^*$; that is, $T(A^*) = T(A) + \alpha\lambda^*$. Thus, every A-interior (B-interior) experiment σ is also A^α-interior. Pick such a σ and a corresponding neighborhood B^ε, and let $f^\alpha := c^\sigma(A^\alpha)$. Importantly, $F^{A^\alpha}(B^\varepsilon)$ contains a full-dimensional subset of F because A^α is an independent menu (since A and B are oriented).

For every α, f^α is in the interior of $F^{A^\alpha}(B^\varepsilon)$. Let $\delta(\alpha) > 0$ denote the radius of the largest open ball around f^α contained in $F^{A^\alpha}(B^\varepsilon)$; call this ball B^α. Clearly, f^α and $\delta(\alpha)$ are continuous in α. Therefore $\delta^* = \min_\alpha \delta(\alpha)$ is well-defined.

Now construct a finite sequence $\alpha(0), \alpha(1), \ldots, \alpha(I)$ such that $\alpha(0) = 0$, $\alpha(I) = 1$, and $d(f^{\alpha(i)}, f^{\alpha(i-1)}) < \delta^*/2$ for all $i = 1, \ldots, I$, where d denotes the standard Euclidean metric. This can be done because f^α is continuous in α. Notice that $f^{\alpha(i)} \in B^{\alpha(i-1)}$ for all $i = 1, \ldots, I$. Thus, $B^{\alpha(i)}$ and $B^{\alpha(i-1)}$ intersect in a full-dimensional region, so that $A^{\alpha(i)}$ and $A^{\alpha(i-1)}$ share a representation. Hence, A and B share a representation. $\square$

Lemma 16 (Face Expansion)**.** *Let A be an independent menu and suppose $\mathcal{N} = \mathcal{N}(A) \cup \{\lambda\}$ for some $\lambda \in S^\Omega_+$. Then there is an independent menu B such that (i) $\mathcal{N}(B) = \mathcal{N}$, and (ii) A and B share a representation.*

Proof. Fix an A-interior experiment σ and an ε-neighborhood B^ε around σ. Without loss of generality, no $s \in \sigma$ is of the form $s = \gamma\lambda$ for any $\gamma > 0$ (if necessary, choose some other $\sigma' \in B^\varepsilon$ and redefine σ to be σ'). Let $f^* := c^\sigma(A)$. Since A is independent, the set $F^A(B^\varepsilon)$ contains a ball of radius δ around f^* for some $\delta > 0$.

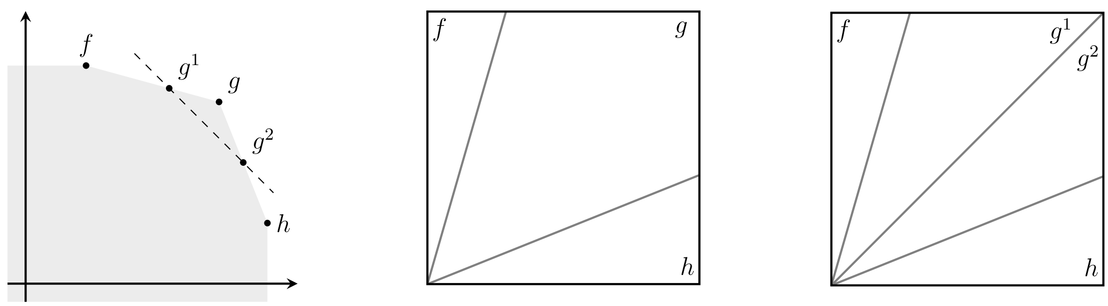
Let $H := \{\lambda' \in \mathbb{R}^\Omega : \lambda \cdot \lambda' = \zeta\}$ denote the (unique) hyperplane with normal λ that intersects the boundary (but not the interior) of $T(A)$. The half-space $H^*(\zeta) := \{\lambda' \in \mathbb{R}^\Omega : \lambda \cdot \lambda' \leq \zeta\}$ below H contains $T(A)$. Shifting H^* toward the origin by a small amount (that is, taking $H^*(\zeta')$ with $\zeta' < \zeta$) and intersecting with $T(A)$ yields a new decision polytope T' where one or more vertices of $T(A)$ are split into multiple vertices. This means that for at least one $f \in A$, the vertex $z^f = U(f) \in T(A)$ is split into vertices $z^{f_1}, \ldots, z^{f^n}$ in T', and the set $S^A(f)$ is divided into convex cones $S(f^i) \subseteq S^A(f)$ where $S(f^i) := \{s \in S : s \cdot z^{f^i} > s \cdot z \ \forall z' \neq z^{f^i}\}$.

By construction, T' has a face with normal λ. By letting $\zeta' \to \zeta$, T' converges to $T(A)$ (in the Hausdorff metric). Thus, if the vertex $z^f \in T(A)$ corresponding to some $f \in A$ is split into $z^{f_1}, \ldots, z^{f^n}$ in T', the coordinates z^{f^i} each converge to z^f as $\zeta' \to \zeta$. Therefore, acts f^i such that $U(f^i) = z^{f^i}$ can be chosen such that $f^i \to f$ as $\zeta' \to \zeta$. Moreover, the acts corresponding to new vertices can be chosen so that the resulting menu B is independent (perturb the constituent lotteries along indifference curves for u if necessary).

Thus, there is a ζ' near ζ for which the corresponding menu B satisfies $d(f^*, c^\sigma(B)) < \delta$; that is, $c^\sigma(B)$ is in the interior of the ball of radius δ around f^*. Since B is independent, F^B contains a ball of radius δ' around $c^\sigma(B)$ for some $\delta' > 0$. Thus, $\dim(F^A \cap F^B) = \dim(F)$, so that A and B share a representation. $\square$

Lemma 17. *Suppose A is a k-menu and $B \subseteq A$ such that $c^e(A) \in B$. There exists an experiment σ such that $c^s(A) \in B$ for all $s \in \sigma$. Moreover, σ may be chosen so that for each $f \in B$, σ contains a signal s^f such that $c^{s^f}(A) = B$.*

Proof. Let $f^e \in B$ denote the act satisfying $c^e(A) \in B$. For each $f \in B \backslash f^e$, pick s^f such that $c^{s^f}(A) = f$; such s^f exist because A is a k-menu. Let $s := \sum_{f \in B \backslash f^e} s^f$, and choose



(a) Utility Space

(b) Signal space for $A = \{f, g, h\}$

(c) Signal space for $B = \{f, g^1, g^2, h\}$

Figure 9: Illustration of Lemma 16. The shaded region in (a) is $T(A)$. T' is formed by clipping off the region above the dashed line, effectively replacing coordinate g with coordinates g^1 and g^2. The region in signal space where g is chosen from $A = \{f, g, h\}$ is divided into regions for g^1 and g^2 in menu $B = \{f, g^1, g^2, h\}$ (in this example, the dashed line is orthogonal to e). If the acts yielding utility coordinates g^1 and g^2 are sufficiently close to g, then A-interior experiments σ yield induced acts $c^\sigma(A)$ and $c^\sigma(B)$ that are close to each other.

$\alpha \in (0, 1)$ such that $e - \alpha s \in S^A(f^e)$. Such an α exists because for small enough α, $e - \alpha s$ is close to $e \in S^A(f^e)$, which is a full-dimensional subset of $S = [0, 1]^\Omega \backslash 0$. Finally, let $\sigma = \{\alpha s^f : f \in B \backslash f^e\} \cup \{e - \alpha s\}$. Since $c^{\lambda t} = c^t$ for all $\lambda > 0$ such that $\lambda t \in S$, it follows that σ is a well-defined experiment satisfying all desired properties. $\square$

Lemma 18. *Suppose U is a k-utility profile and U' is an ℓ-utility profile such that $T = T(U)$ and $T' = T(U')$ satisfy $\frac{1}{W}e \in \mathcal{N}(T) \cap \mathcal{N}(T')$. For each choice of A and B such that $U = U(A)$ and $U' = U(B)$, there exists an N-utility profile U^* and a $\lambda \in \mathbb{R}^\Omega$ such that:*

(i) $U \cup U^*$ is a $(k + N)$-utility profile and $U' \cup (U^* + \lambda)$ is a $(\ell + N)$-utility profile,

(ii) There is a $z \in U^*$ such that $e \in S^{U \cup U^*}(z)$ and $e \in S^{U' \cup (U^* + \lambda)}(z + \lambda)$, and

(iii) If $U^* = U(A^*)$ and $U^* + \lambda = U(B^*)$, then A inherits a representation from $A \cup A^*$ and B inherits a representation from $B \cup B^*$.

Proof. Let A and B satisfy $U = U(A)$ and $U' = U(B)$. Choose an A-interior experiment σ and a corresponding neighborhood B^ε, and a B-interior σ' with neighborhood $B^{\varepsilon'}$. As in the proof of Lemma 12, the half-spaces corresponding to signals $s \in \hat{\sigma} \in B^\varepsilon$ passing through the point $U(f^s)$ (where $f^s = c^s(A)$) intersect to form a space $T^*(A)$ such that $T(A) \subseteq T^*(A)$. Moreover, $T^*(A) \backslash T(A)$ contains a full-dimensional subset of $\mathbb{R}^\Omega$ near the face of $T(A)$ with normal e because every $s \in \hat{\sigma} \in B^\varepsilon$ is bounded away from e. In other words, $T^*(A) \backslash T(A)$ contains a full-dimensional subset of the region above the hyperplane corresponding to this face. A similar argument yields a region $T^*(B)$ for which analogous statements hold.

Thus, there is a $\delta > 0$ such that both $T^*(A)\backslash T(A)$ and $T^*(B)\backslash T(B)$ contain an open ball of radius δ. Letting D^A denote such a ball in $T^*(A)\backslash T(A)$ and D^B the ball in $T^*(B)\backslash T(B)$, it follows that $D^B = D^A + \lambda$ for some $\lambda \in \mathbb{R}^\Omega$.

The profile U^* is constructed as follows. First, pick a point $z^1 \in D^A$. Then $z^1 + \lambda \in D^B$. By our choice of D^A and D^B, we have that $T(U \cup \{z^1\})$ is a $(k+1)$-polytope such that $e \in S^{U \cup \{z^1\}}(z^1)$; that is, if some act f^1 satisfies $U(f^1) = z^1$, then $c^e(A \cup \{f^1\}) = f^1$. Since this is a strict preference, there is in fact a full-dimensional, convex set of signals s such that $c^s(A \cup \{f^1\}) = f^1$, and e belongs to the interior of this set. Similar statements hold for $B \cup \{g^1\}$ for any g^1 such that $U(g^1) = z + \lambda$. Therefore, there is a full-dimension set of signals s such that $c^s(A \cup \{f^1\}) = f^1$ and $c^s(B \cup \{g^1\}) = g^1$. Call the set of all such s the *support* of z^1.

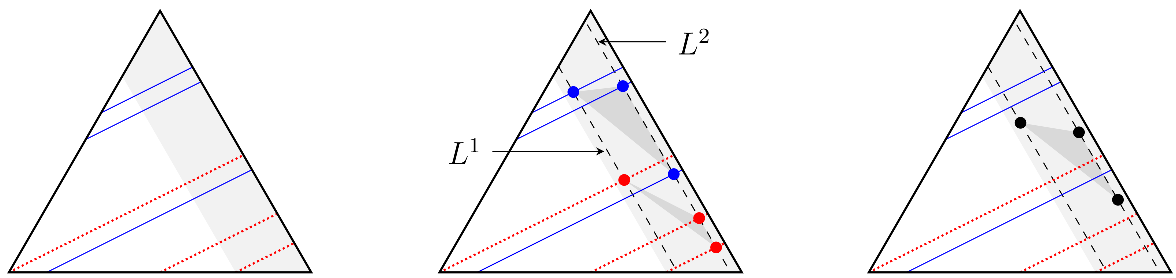
We now proceed by induction. Suppose $U^* = \{z^1, \ldots, z^n\} \subseteq D^A$ such that each $z \in U^*$ has full-dimensional support. That is, for any A^* such that $U(A^*) = U^*$ and each $f \in A^*$, the set $S^z = S^{A \cup A^*}(f) \cap S^{B \cup (A^* + \lambda)}(g)$ has full dimension, where $g \in B^*$ satisfies $U(g) = U(f) + \lambda$. Pick an s in the interior of S^z such that $s^z \neq e$ and consider the hyperplane $H(s; z)$ with normal s passing through z. Now pick a point $z^{n+1} \in H(s; z) \backslash z$; if z^{n+1} is sufficiently close to z, then $z^{n+1} \in D^A$, $T(U \cup U^* \cup \{z^{n+1}\})$ is a $(k+n+1)$-polytope, and $T(U' \cup (U^* \cup \{z^{n+1}\} + \lambda))$ is an $(\ell + n + 1)$-polytope. Moreover, z^{n+1} has full dimensional support.

The resulting set $U^* = \{z^1, \ldots, z^N\}$ clearly satisfies (i) and (ii). For (iii), note that our original choice of D^A and D^B guarantees that for all $s \in \hat{\sigma} \in B^\varepsilon$, $c^s(A \cup A^*) = c^s(A)$ and $s' \in \hat{\sigma}' \in B^{\varepsilon'}$ implies $c^{s'}(B \cup B^*) = c^{s'}(B)$. Thus, $F^A(B^\varepsilon) \subseteq F^{A \cup A^*}$ and $F^B(B^\varepsilon) \subseteq F^{B \cup B^*}$, so that $\dim(F^A) \leq \dim(F^{A \cup A^*})$ and $\dim(F^B) \leq \dim(F^{B \cup B^*})$. □

Lemma 19. *Suppose $U, U' \subseteq (0,1)$ are sets of cardinality N. There exist $P, Q \subseteq \Delta X$ and a bijection $\varphi : P \to Q$ such that φ is oriented, $U = \{u(p) : p \in P\}$, and $U' = \{u(q) : q \in Q\}$.*

Proof. Figure 10 illustrates the idea of the proof. Consider the indifference curves (hyperplanes) in ΔX corresponding to the utilities in $U \cup U'$. There is an edge E of ΔX such that each of these planes intersects the (relative) interior of E. Specifically, E is any edge connecting lotteries δ_b and δ_w for any choice of $b, w \in X$ such that $u(b) \geq u(x) \geq u(w)$ for all $x \in X$. Since each utility level is interior, it can be expressed as a non-degenerate mixture of $u(b)$ and $u(w)$, forcing the associated hyperplane to intersect the relative interior of E. Parallel to this edge is an interior line L passing through (the interior of) each hyperplane, so that in fact there is an $\varepsilon > 0$ such that every parallel ε perturbation of L passes through each hyperplane. Let $B \subseteq \Delta X$ denote the region spanned by these perturbations; clearly, B has dimension equal to that of ΔX (namely, $N-1$).

Now pick $N-1$ lines $L^1, \ldots, L^{N-1}$ in B, each parallel to L, such that the convex hull of $\{L^1, \ldots, L^{N-1}\}$ has dimension $N-1$. Rank the numbers in $u^i \in U$ so that $u^1 > u^2 > \ldots >$



(a) Utility Levels
(b) P (blue dots) and Q (red dots)
(c) P^α, $\alpha = 0.5$

Figure 10: Illustration of Lemma 19. The solid (blue) lines are the utility levels for U, and the dotted (red) lines are utility levels for U'. The shaded region in (a) is the region $B \subseteq \Delta X$ referenced in the proof. With this construction, every set P^α is affinely independent.

u^N. For $i = 1, \ldots, N-1$, let p^i be the (unique) intersection of L^i and the indifference plane for utility u^i, and let p^N be the unique intersection of L^{N-1} with the indifference plane for utility u^N. Observe that $\{p^1, \ldots, p^{N-1}\}$ lie on a hyperplane H in ΔX and that p^N is not in the affine hull of H because L^{N-1} passes through H at a single point (p^{N-1}) while p^N lies at a different point on L^{N-1}. Thus, $P = \{p^1, \ldots, p^N\}$ is affinely independent.

Using the same lines $L^1, \ldots, L^{N-1}$ and the same rank-based construction for U' yields an affinely independent set $Q = \{q^1, \ldots, q^N\}$ where $u(q^1) > \ldots > u(q^N)$.

Now consider $P^\alpha := \{(1-\alpha)p^i + \alpha q^i : i = 1, \ldots, N\}$. Observe that $(1-\alpha)u(p^i) + \alpha u(q^i) > (1-\alpha)u(p^{i+1}) + \alpha u(q^{i+1})$ for all $i = 1, \ldots, N-1$ because $u(p^i) > u(p^{i+1})$ and $u(q^i) > u(q^{i+1})$. Notice also that $(1-\alpha)p^i + \alpha q^i$ is on line L^i ($i = 1, \ldots, N-1$) and $(1-\alpha)p^N + \alpha q^N$ is on L^{N-1}. Thus, by the same argument, P^α is affinely independent. Hence, the map $\varphi : P \to Q$ given by $\varphi(p^i) = q^i$ ($i = 1, \ldots, N$) is oriented. $\square$

Lemma 20. *If A and B are independent, then A and B share a representation.*

Proof. By Lemma 16, we may assume that $e \in \mathcal{N}(A)$ and $e \in \mathcal{N}(B)$. Then, by Lemma 18, there is a utility profile U and a $\lambda \in \mathbb{R}^\Omega$ such that if $U = U(A^*)$ and $U' := U + \lambda = U(B^*)$, then A and $A' := A \cup A^*$ share a representation, and B and $B' := B \cup B^*$ share a representation. In fact, by Lemma 17, A' shares a representation with A^* provided A^* is independent. Similarly, B' shares a representation with B^* provided B^* is independent. Therefore, it will suffice to find independent menus A^* and B^* such that $U = U(A^*)$, $U' = U(B^*)$, and such that A^* and B^* share a representation.

To do so, choose a state ω and apply Lemma 19 to the sets $U_\omega := \{z_\omega : z \in U\}$ and $U'_\omega := \{z'_\omega : z' \in U'\}$ to get affinely independent sets $P_\omega := \{p^z_\omega : z \in U\}$ and $Q_\omega := \{q^{z'}_\omega : z' \in U'\}$ such that $u(p^z_\omega) = z_\omega$ and $u(q^{z'}_\omega) = z'_\omega$ for all $z \in U$ and $z' \in U'$ (if necessary, apply a small perturbation to U and U' in order to get N distinct utility values in U_ω for each ω, and N

distinct utility values in U'_ω for all ω). Repeating this for each ω yields acts $f^z := (p^z_\omega)_{\omega\in\Omega}$ and $g^{z'} := (q^{z'}_\omega)_{\omega\in\Omega}$ for each $z \in U$ and $z' \in U'$. Then $A^* := \{f^z : z \in U\}$ and $B^* := \{g^{z'} : z' \in U'\}$ are oriented, so that by Lemma 15, A^* and B^* share a representation. □

Lemma 21. *There is a unique, linear $L^* : F \to \mathbb{R}$ such that, for all k-menus A, the function $\sigma \mapsto L^*(c^\sigma(A))$ represents $\succsim^A$ on $\sigma \in \mathcal{E}^c(A)$.*

Proof. By Lemma 20, all independent menus share a representation. This means there is a unique linear $\succsim$ on F that agrees with each relation $\succsim^B$ where B is independent. This $\succsim$ also agrees with $\succsim^A$ since every k-menu inherits a representation from an independent menu (Lemma 13). To construct L^*, choose any independent menu A and consider the linear representation $W^A : F^A \to \mathbb{R}$ constructed at the start of Step 1. Since F^A has full dimension, W^A has a unique linear extension to F. Take L^* to be this extension. □

Proof of Theorem 1

Theorem 1A. *Suppose c has a Bayeisan Representation. Let $\dot{\succsim} = (\dot{\succsim}^A)_{A\in\mathcal{A}}$ where $\dot{\succsim}^A$ is the restriction of $\succsim^A$ to $\mathcal{E}^c(A)$. Then $(\dot{\succsim}, c)$ satisfies Axioms 1.1–1.6 if and only if there exists a full-support $\nu \in \Delta\Omega$ and a non-constant utility index $v : X \to \mathbb{R}$ such that, for all $A \in \mathcal{A}$ and all $\sigma, \sigma' \in \mathcal{E}^c(A)$,*

$$\sigma \succsim^A \sigma' \Leftrightarrow \sum_{\omega\in\Omega} \nu_\omega \sum_{s\in\sigma} s_\omega v(c^s_\omega(A)) \geq \sum_{\omega\in\Omega} \nu_\omega \sum_{s'\in\sigma'} s'_\omega v(c^{s'}_\omega(A)).$$

Moreover, ν is unique and v is unique up to positive affine transformation.

Proof. Let L^* be the linear representation given by Lemma 21 and let $\sigma, \sigma' \in \mathcal{E}^c(A)$. Then there is a submenu $A' \subseteq A$ that is a k-menu (for some k) such that $c^\sigma(A) = c^\sigma(A')$ and $c^{\sigma'}(A) = c^{\sigma'}(A')$). By the Consistency axiom, $\sigma \succsim^A \sigma'$ if and only if $\sigma \succsim^{A'} \sigma'$. Thus, $\sigma \succsim^A \sigma'$ if and only if $L^*(c^\sigma(A)) \geq L^*(c^{\sigma'}(A))$.

By the Non-Degeneracy axiom, L^* must be non-constant; otherwise, by the previous paragraph, every $\succsim^A$ assigns indifference among all experiments in $\mathcal{E}^c(A)$. Thus, by Lemma 8, $\succsim^{A^*}$ (uniquely) extends to $\succsim$ on F (where A^* is the symmetric menu constructed in Step 1), and $\succsim$ satisfies all of the Anscombe-Aumann axioms, including Non-Degeneracy. Thus, $\succsim$ has an expected utility representation with a unique ν and a unique (up to positive affine transformation) utility index v. Since L^* is a linear representation for $\succsim$, it follows that the expected utility representation holds for all menus $\succsim^A$ on $\mathcal{E}^c(A)$. □

Theorem 1B. *Suppose c has a Bayesian Representation. Then $(\succsim, c)$ satisfies Axioms 1.1–1.7 if and only if it has a Value of Information Representation. Moreover, ν is unique and*

v is unique up to positive affine transformation.

Proof. First, we verify that existence of a Value of Information Representation implies Axiom 1.7. Let A^n be a sequence of menus giving rise to a stable selection $(s, A) \mapsto f^s(A) \in \Delta c^s(A)$. Then $A^n \to^c A$ and $f^s(A) = c^s(A^\infty)$ for all s. Take $\hat{A} = A^*$ constructed in Step 1 above. Choose any $\sigma^* \in \mathcal{E}^c(\hat{A})$ such that $f^* := c^{\sigma^*}(\hat{A})$ is in the interior of F^{A^*}. Let $\sigma, \sigma' \in \mathcal{E}$. Then there exists $\alpha \in (0,1)$ and $\hat{f}, \hat{g} \in F^{A^*}$ such that $\hat{f} = \alpha c^\sigma(A^\infty) + (1-\alpha)f^*$ and $\hat{g} = \alpha c^{\sigma'}(A^\infty) + (1-\alpha)f^*$. Since $\hat{f}, \hat{g} \in F^{A^*}$, there exists $\hat{\sigma}, \hat{\sigma}' \in \mathcal{E}^c(\hat{A})$ such that $\hat{f} = c^{\hat{\sigma}}(\hat{A})$ and $\hat{g} = c^{\hat{\sigma}'}(\hat{A})$. By the Value of Information Representation at A, $\sigma \succsim^A \sigma'$ if and only if the expected utility (under ν and v) of act $c^\sigma(A^\infty)$ is greater than or equal to that of act $c^{\sigma'}(A^\infty)$. Since $\alpha \in (0,1)$, this holds (by the Value of Information Representation at $\hat{A}$) if and only if $\hat{\sigma} \succsim^{\hat{A}} \hat{\sigma}'$, verifying Axiom 1.7.

To see that Axioms 1.1–1.7 imply existence of a Value of Information Representation, first apply Theorem 1A to establish unique ν and v such that the desired representation holds on $\mathcal{E}^c(A)$ for all A. To construct a stable selection, let $A \in \mathcal{A}$ and invoke Axiom 1.7 to get a sequence $A^n \to^c A$ such that $\succsim^{A^n \to^c} A$. Then, for all s, $c^s(A^n)$ converges to a point in the convex hull of A, and if $c^s(A)$ is single-valued, then $c^s(A^n)$ converges to the sole member of $c^s(A)$ (this holds because, by Theorem 1A, the desired representation with menu-independent ν and v holds on $\mathcal{E}^c(A^n)$). Thus, $(s, A) \mapsto c^s(A^\infty)$ is a stable selection. Employing $\hat{A} = A^*$ from Step 1 of the proof in a similar fashion to the argument above, Axiom 1.7 implies that $\sigma \succsim^A \sigma'$ if and only if the expected utility (under ν and v) of $c^\sigma(A^\infty)$ is weakly greater than that of $c^{\sigma'}(A^\infty)$. □

Theorem 1C. *Suppose c has a Bayesian Representation. Then $(\succsim, c)$ satisfies Axioms 1.1–1.6 and 1.7' if and only if, for all $A \in \mathcal{A}$, $\succsim^A$ is represented by the function*

$$\overline{V}^A(\sigma) := \max \sum_{\omega \in \Omega} \nu_\omega \sum_{s \in \sigma} s_\omega v(f^s_\omega) \quad \text{subject to} \quad f^s \in c^s(A).$$

Moreover, ν is unique and v is unique up to positive affine transformation.

Proof. First, we verify that existence of the representation implies Axiom 1.7'. Let $V : F \to \mathbb{R}$ be a utility function for acts using ν and v from the representation; that is, $V(f)$ is the expected utility of f given prior ν and utility index v. Every sequence $A^n \to^c A$ gives rise to a selection. So, if $A^n \to^c A$, then $V(c^\sigma(A^\infty))$ is well-defined and denotes the expected utility of the "induced act" under the selection implied by $A^n \to^c A$.

If $\overline{V}^A(\sigma) \geq \overline{V}^A(\sigma')$, then the best-case selection for σ is weakly better than the best-case selection for σ'. So, if $A^n \to^c A$, then the implied selection at σ' gives rise to an act g such that $\overline{V}^A(\sigma) \geq V(g) = V(c^{\sigma'}(A^\infty))$. It will suffice to show that there is a sequence $B^n \to^c A$

such that $V(c^\sigma(B^n)) = \overline{V}^A(\sigma)$ (in other words, that there is a sequence $B^n \to^c A$ for which the implied selection gives the best-case at σ). Then, employing $\hat{A} = A^*$ in a similar fashion as in the proof of Theorem 1B, one can find the desired $\hat{\sigma}, \hat{\sigma}'$, and σ^*.

The sequence $(B^n)_{n=1}^\infty$ is constructed by specifying utility coordinates for the acts (for Receiver); once a convergent sequence of utility coordinates are specified, an associated convergent sequence of acts (giving rise to those utilities) can be found. So, let $A \in \mathcal{A}$. Enumerate the acts in A as $f^1, \ldots, f^K$. Choose sequences $\varepsilon_n^k \to 0$ such that $\varepsilon_n^k > 0$ for all n and $k > \ell \Rightarrow \varepsilon_n^k > \varepsilon_n^\ell$ for all n. For each $f^k \in A$, choose an act $\tilde{f}^k$ such that $u(\tilde{f}^k) = \frac{\nu}{\mu} v(f^k)$, where $u(\tilde{f}^k) := (\mu_\omega u(f_\omega^k))_{\omega\in\Omega}$ and $\frac{\nu}{\mu} v(f^k) := \left(\frac{\nu_\omega}{\mu_\omega} v(f_\omega^k)\right)_{\omega\in\Omega}$. Choose a sequence $\lambda_n \to 0$ such that $\lambda_1 = 1$ and $\lambda_n > 0$ for all n. For each n, choose $f^{k,n}$ such that $u(f^{k,n}) := (1-\lambda_n)u(f^k) + \lambda_n[u(f^{k,n}) + \varepsilon_n^k]$. It follows that $u(f^{k,n}) \to u(f^k)$ as $n \to \infty$. We may also assume that $u(f^{k,n})$ is in the range of u for all n (if necessary, choose a constant $\alpha^* \in (0,1)$ and mix each $f^{k,n}$ with some f^* such that $u(f_\omega^*)$ is in the interior of the range of u). Thus, we can choose acts $f^{k,n} \to f^k$ satisfying all of these properties, and let $A^n := \{f^{k,n} : f^k \in A\}$. It is now straightforward (but tedious) to verify that (i) if $c^s(A) = f^k$, then $c^s(A^n) \to f^k$; (ii) if $c^s(A)$ is multi-valued and V^s has a unique maximum on $c^s(A)$ (where $V^s : F \to \mathbb{R}$ is Sender's expected utility of acts using v and the Bayesian posterior ν^s), then $c^s(A)$ converges to Sender's preferred act in $c^s(A)$; and (iii) if $c^s(A)$ is multi-valued and V^s does not attain a unique maximum on $c^s(A)$, then $c^s(A^n)$ converges to the f^k (with largest k) among those acts in $c^s(A)$ that maximize V^s. Thus, $A^n \to^c A$ and the implied selection satisfies $V(c^\sigma(A^\infty)) = \overline{V}^A(\sigma)$, as desired. Intuitively, the idea of the construction is to perturb the menu A in the direction of Sender's preferences. Since $\lambda_n \to 0$, this perturbation only has an effect in the limit if Receiver is indifferent between two or more acts at some signal realization. The sequences ε_n^k are needed to handle situations where both Sender and Receiver are tied at some signal realization; the enumeration $f^1, \ldots, f^K$ is used to break those ties.

To see that Axioms 1.1–1.6 and 1.7' imply the desired representation, once again invoke Theorem 1A to establish the representation on $\mathcal{E}^c(A)$ for all A. Then Axiom 1.7' (along with similar constructions as above) immediately imply that $\sigma \succsim^A \sigma' \Leftrightarrow \overline{V}^A(\sigma) \geq \overline{V}^A(\sigma')$. □

B Proof of Theorem 2

I prove that if c satisfies Axioms 2.1—2.2, then c has a Bayesian representation (μ, u) (the converse is straightforward). By part (i) of Axiom 2.1, each c^s has a rationalizing preference relation: there exists a (unique) complete and transitive binary relation $\succsim^s$ on F such that, for all $A \in \mathcal{A}$, $c^s(A) = \{f \in A : f \succsim^s g \ \forall g \in A\}$. By part (ii) of Axiom 2.1, $\succsim^s$ is non-degenerate: $\exists f, g \in F$ such that $f \succ^s g$, where $\succ^s$ denotes the strict part of $\succsim^s$.

Lemma 22. *If $f \succ^s g$ and $\alpha \in (0,1)$, then $\alpha f + (1-\alpha)h \succ^s \alpha g + (1-\alpha)h$ for all $h \in F$.*

Proof. Let $A = \{f, g\}$ and $B = \{h\}$. Since $f \succ^s g$, we have $c^s(A) = \{f\}$. Thus, $\alpha c^s(A) + (1-\alpha)c^s(B) = \{\alpha f + (1-\alpha)h\}$, so that (by part (iii) of Axiom 2.1) $c^s(\alpha A + (1-\alpha)B) = \{\alpha f + (1-\alpha)h\}$. Since $\alpha g + (1-\alpha)h \in \alpha A + (1-\alpha)B$, it follows that $\alpha f + (1-\alpha)h \succ^s \alpha g + (1-\alpha)h$. □

Lemma 23. *Each $\succsim^s$ is continuous (that is, weak contour sets are closed).*

Proof. Observe that c^s is closed-valued (because $c^s(A)$ is finite for all $A \in \mathcal{A}$) and upper-hemicontinuous (part (iv) of Axiom 2.1). Thus, c^s has the closed-graph property: if $A^n \to A$, $f^n \to f$, and $f^n \in c^s(A^n)$ for all n, then $f \in c^s(A)$.

To see that upper contour sets of $\succsim^s$ are closed, fix g and suppose $f^n \to f$ where $f^n \succsim^s g$ for all n. Then $f^n \in c^s(\{g, f^n\})$ for all n. Clearly, $\{g, f^n\} \to \{g, f\}$. Thus, by the closed-graph property, $f \in c^s(\{g, f\})$, so that $f \succsim^s g$.

For the lower contour sets, fix g and suppose $f^n \to f$ where $g \succsim^s f^n$ for all n. Letting $g^n = g$ for all n, it follows that $g^n \in c^s(\{g, f^n\})$ for all n. Clearly, $\{g, f^n\} \to \{g, f\}$ and $g^n \to g$. Thus, by the closed-graph property, $g \in c^s(\{g, f\})$, so that $g \succsim^s f$. □

Lemma 24. *If $p[\omega]h \succsim^s q[\omega]h$ and $s_\omega, s'_{\omega'} > 0$, then $p[\omega']h' \succsim^{s'} q[\omega']h'$ for all $h' \in F$.*

Proof. Let $L = \{p, q\}$. Since $p[\omega]h \succsim^s q[\omega]h$, we have $p[\omega]h \in c^s(L[\omega]h)$. Thus, by part (v) of Axiom 2.1, we have $p[\omega']h' \in c^{s'}(L[\omega']h')$, so that $p[\omega']h' \succsim^{s'} q[\omega']h'$. □

By Lemmas 22, 23, and 24, each $\succsim^s$ satisfies the Anscombe-Aumann axioms and, hence, can be represented by standard expected utility with a prior μ^s and (non-constant) utility index u^s. The state independence axiom expressed by Lemma 24 implies that μ^e (where $e = (1, \ldots, 1) \in S$) has full support, and that for all $s, s' \in S$, u^s is a positive affine transformation of $u^{s'}$. Thus, we may assume $u^s = u := u^e$ for all s. To complete the proof, we verify that μ^s is the Bayesian posterior of $\mu := \mu^e$ conditional on s; that is, that $f \succsim^s g \Leftrightarrow \sum_\omega u(f_\omega)s_\omega\mu^e_\omega \geq \sum_\omega u(g_\omega)s_\omega\mu^e_\omega$. Notice that $f \succsim^s g$ if and only if $ef + (1-e)h \succsim^s eg + (1-e)h$, which (by Axiom 2.2) holds if and only if $sf + (1-s)h \succsim^e sg + (1-s)h$. By the expected utility representation for $\succsim^e$, $sf + (1-s)h \succsim^e sg + (1-s)h \Leftrightarrow \sum_\omega [s_\omega u(f_\omega) + (1-s_\omega)u(h_\omega)]\mu^e_\omega \geq \sum_\omega [s_\omega u(g_\omega) + (1-s_\omega)u(h_\omega)]\mu^e_\omega \Leftrightarrow \sum_\omega u(f_\omega)s_\omega\mu^e_\omega \geq \sum_\omega u(g_\omega)s_\omega\mu^e_\omega$, as desired.

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