Dynamic (In)Consistency and the Value of Information

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Abstract

This paper develops a revealed-preference model of information disclosure. A sender ranks information structures (Blackwell experiments) knowing that a receiver uses the information to select an action affecting them both. The two decision makers may differ in their utility functions and/or prior beliefs, yielding a model of dynamic inconsistency when the sender and receiver represent the same individual at two points in time. I take as primitive, for each menu of acts, (i) a preference ordering over all Blackwell experiments (the sender's preference for information), and (ii) a correspondence indicating the receiver's signal-contingent choices from the menu. I derive axiomatic representation theorems characterizing the sender as a sophisticated planner and the receiver as a Bayesian information processor, and show that all parameters can be uniquely identified from the sender's preferences for information. I also establish a series of results characterizing common priors, common utility functions, and intuitive measures of disagreement for these parameters—all in terms of the sender's preferences for information.

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1 Introduction

People regularly choose among different sources of information. For example, they choose which newspapers to read, web sites to browse, or experts to consult. A distinguishing feature of information is that its value to a standard rational agent is purely instrumental: it is valued only to the extent that it improves choice among risky alternatives. Consequently, the value of information depends on many variables, including the set of feasible alternatives and the utilities and prior beliefs of the individual. How does a decision maker's value of information vary with these parameters, and what can be inferred from his preference for information?

I approach these questions from a revealed-preference perspective by formulating a model of information disclosure with two decision makers: a sender (DM1) and receiver (DM2). The sender ranks information structures knowing the receiver uses the information to choose a risky alternative from some set. I assume the receiver's signal-contingent choices are observable. A key feature of the model, however, is that the sender's preference for information—not his ranking of acts or menus—is an observable primitive. This captures the idea that the sender can control the information available to the receiver, but not the set of actions. Thus, information is more valuable to the sender when, on average, it guides the receiver toward actions that are more attractive to the sender.

Both the sender and receiver are expected utility maximizers. However, they may differ in their utility functions or prior beliefs. This enables two interpretations of the model. In the *persuasion* interpretation, the sender and receiver represent distinct individuals. Hence, the framework provides a decision-theoretic foundation for "Bayesian persuasion" models (Kamenica and Gentzkow, 2011). In the *behavioral* interpretation, the sender and receiver represent the same individual at two points in time, yielding a model of dynamically inconsistent behavior. As is well-known, sophisticated, dynamically inconsistent individuals value commitment power.[1] Here the sender lacks hard commitment power in that he cannot restrict the set of actions available to the receiver. Instead, he commits to revealing the signal generated by the chosen information structure. Hence, informational choice offers an alternative form of commitment power, and preferences for information reflect preferences for commitment.

To illustrate the main ideas, as well as the behavioral interpretation, consider an individual who must decide whether to consume a particular dessert (action D) or not (action $\neg D$). The dessert contains an ingredient that is either unhealthy (state G) or very unhealthy (state B). In period 1, before the decision is to be made, the individual is health-conscious:

[1] See, for example, Strotz (1955).

	G	B
D	0	-2
$\neg D$	1	1

(a) Utilities v

	G	B
D	2	0
$\neg D$	1	1

(b) Utilities u

he prefers not to consume the dessert regardless of the state of the world (preferences v above). He recognizes, however, that he may succumb to temptation when confronted with the choice: his future-self prefers to consume the dessert in state G but to refrain in state B (preferences u).

Lacking hard commitment power, the period-1 individual (DM1) attempts to influence future choice through careful exposure to information. For example, he may consult a specialist who correctly reveals the true state, or browse web sites containing imperfect information about the state. If he acquires sufficient evidence of state B, his period-2 self (DM2) will refrain from consuming the dessert despite the lack of hard commitment power.

Differences between first- and second-period utility functions induce non-trivial preferences for information. If, for example, both selves assign prior probability 2/3 to state G, then DM1 prefers perfect information over no information: perfect information results in choice $\neg D$ with probability 1/3, while no information results in choice D with probability 1. However, perfect information is not ideal for DM1. Consider the following information structure (denoted σ):

	s	t
G	1/4	3/4
B	1	0

This information structure generates signal s in state B, while in state G it generates s with probability 1/4 and t with probability 3/4. Under Bayesian updating, DM2 chooses D at signal t and $\neg D$ at s. Thus, DM1 achieves a higher expected payoff from σ than from perfect information, so that his preference for information violates the Blackwell (1951, 1953) information ordering. In a similar fashion, non-common priors also lead to violations of the Blackwell ordering.[2] A key finding of this paper is that such violations are very informative and that, in fact, the sender's preferences for information fully reveal the priors and utilities of both decision makers.

[2]Heterogeneous priors can be interpreted as a different source of temptation. In this example, both decision makers could hold utility function u while the second-period prior is skewed in favor of state G. Thus, the effect of temptation is to become biased or delusional in favor of state G, making the dessert seem more attractive. The decision maker knows himself well enough to anticipate his behavior in such choice environments.

In the representation, the receiver selects among *acts* (Anscombe and Aumann, 1963): profiles $f = (f_\omega)_{\omega\in\Omega}$ assigning lotteries $f_\omega \in \Delta X$ to states of the world $\omega \in \Omega$, where X and Ω are finite sets of outcomes and states, respectively. Information structures take the form of Blackwell experiments associating probability distributions over a set of signals to different states of the world. As illustrated above, a Blackwell experiment is a matrix σ where each column represents a signal and each row ω (one for each state) is a probability distribution over the signals.

The receiver is characterized by a family of *signal-contingent choice correspondences* c^s. A *signal* is a profile $s = (s_\omega)_{\omega\in\Omega}$ of entries from $[0,1]$ (not all zero). In other words, s coincides with a column from some experiment and the entries of s represent likelihoods of the signal being generated in different states of the world. For a signal s and *menu* A (a finite set of acts), $c^s(A) \subseteq A$ is the set of acts chosen by the receiver after observing s.

The sender is characterized by a family of preference relations $\succsim^A$ indexed by menus A. Each $\succsim^A$ is an ordering of the set of all Blackwell experiments and represents the sender's *preference for information* when the set of alternatives is A. The statement $\sigma \succsim^A \sigma'$ means the sender expects (on average) higher payoffs from σ than σ', given that his outcome is determined by the receiver's signal-contingent choices from A.

The representation theorem is divided into two parts: one for the sender (Theorem 1) and one for the receiver (Theorem 2). For the receiver, the goal is to rationalize choices c^s as expected utility maximization under some utility index u, prior μ (full support), and Bayesian updating. The rich space of signals employed here admits a novel characterization of such behavior. The key axiom, *Bayesian Independence*, expresses an equivalence between scaling utilities and scaling signal likelihoods. Combined with a new continuity axiom and other standard axioms, this characterizes the receiver as a Bayesian information processor (and expected utility maximizer).

For the sender, each relation $\succsim^A$ is represented by expected utility maximization under prior ν, utility index v, and correct forecasting of the receiver's choices. Let $f^s(A)$ denote the act chosen from A by the receiver when signal $s \in \sigma$ realizes.[3] The sender assigns utility

$$V^A(\sigma) = \sum_\omega \nu_\omega \sum_{s\in\sigma} s_\omega v(f^s_\omega(A)) \tag{1}$$

to σ, where ν_ω is the sender's prior probability of state ω and $v : X \to \mathbb{R}$ is his utility index.[4] This is analogous to an indirect utility function for the sender in Bayesian Persuasion models

[3]The statement '$s \in \sigma$' means s is a column of σ (s_ω is the column's entry for row ω). Assume, for now, that the receiver is not indifferent between two or more acts of A at any signal $s \in \sigma$.

[4]Abusing notation slightly, let $v(p) := \sum_x v(x)p(x)$ for lotteries $p \in \Delta X$.

(Kamenica and Gentzkow, 2011).

The axioms characterizing representation (1) employ both informational preferences $\succsim^A$ and signal-contingent choices c^s. Familiar (von Neumann-Morgenstern) Independence and Continuity axioms are defined using an appropriate mixture operation on the space of experiments, and the (Anscombe-Aumman) State Independence axiom is expressed using both preferences $\succsim^A$ and choices c^s. Thus, with one exception, the representation rests on standard axioms adapted to the present setting. The key axiom, *Consistency*, is the exception; it places minimal restrictions on how preferences $\succsim^A$ may vary as the menu A changes. The main challenge is to show that under *Consistency*, the prior ν and utility index v of the sender are not menu-dependent.

The combined representation theorem (Theorem 3) establishes uniqueness of all parameters (ν, v, μ, and u) given both the sender's preferences for information and the receiver's signal-contingent choices. It turns out, however, that all parameters can be identified from the sender's preferences for information (Theorem 4). This is surprising because information structures are only indirectly related to the choices made by the receiver and, hence, the outcomes that agents actually care about. Heterogeneous priors and utilities further complicate matters. In section 5, I first show how the priors can be elicited from the sender's preferences. Then, given the priors, the utility indices can be identified.

Preferences for information can also be used to make comparisons between the attributes of the sender and receiver. I show in section 6 that the sender and receiver have a common utility index if and only if the sender prefers full disclosure (revealing the true state) in all menus. This provides a simple, testable characterization of common utilities that holds independently of whether there is a common prior. In a similar spirit, common priors can be tested independently of the utility indices: $\nu = \mu$ if and only if the sender's preferences are monotone with respect to the Blackwell information ordering in a class of menus called *bets*.[5] Thus, if A is a bet, then $\succsim^A$ is either a completion of the Blackwell ordering or a completion of the opposite (inverse Blackwell) ordering. Combined, these results show that the sender and receiver share common priors and utilities if and only if the sender's preferences satisfy the Blackwell ordering in all menus. Hence, in the behavioral interpretation, dynamic consistency is equivalent to the Blackwell ordering.

Finally, section 6 also develops methods to assess the degree of separation between the attributes of the sender and receiver. In particular, the utility functions v and u exhibit "more agreement" if the sender prefers full disclosure in a larger set of bets, while the priors ν and μ exhibit more agreement if the sender's preferences exhibit fewer violations of Blackwell monotonicity at "extremes"—pairs of experiments where at least one is reasonably

[5] A bet is a menu $A = \{f, g\}$ where there exist lotteries $p, q \in \Delta X$ such that $f_\omega, g_\omega \in \{p, q\}$ for all ω.

informative.

These results illustrate the power and applicability of information structures as objects of choice. Preferences for information may seem rather abstract, but it is not difficult to see how individuals reveal such preferences in different environments. For example, many online newspapers allow subscribers to customize their news feeds by selecting categories (sports, finance, politics, etc) about which they will be informed of new developments, while online retailers enable custom tailoring of information about new products or services. By customizing such news feeds, individuals reveal what type of information they consider to be the most valuable. Laboratory settings, of course, provide another setting where preferences for information can be directly elicited. This paper does not carry out any empirical or experimental exercises, but demonstrates that informational choice may be a valuable resource for analysts interested in testing models or identifying parameters.

Finally, note that preferences for information are a natural primitive in both interpretations of the model. In Bayesian Persuasion settings, informational preferences of the sender (together with signal-contingent choices of the receiver) are the most an analyst can hope to observe. In the behavioral interpretation, informational choice offers an effective form of commitment power. Hard commitment opportunities are relatively rare compared to the abundance of available information sources. Thus, while an individual might not be able to avoid encountering tempting alternatives, he may be able to resist temptation when it arrives by selectively paying attention to information sources—in particular, ones that are more likely to make tempting alternatives seem less appealing.

1.1 Related Literature

In general, this paper is related to the rapidly growing literature on information disclosure with sender commitment power.[6] This literature was initiated by Kamenica and Gentzkow (2011) (henceforth KG) and Rayo and Segal (2010). My model is most closely related to the framework of KG, where a sender chooses an experiment and a receiver takes an action after observing a signal generated by the experiment. Building on techniques of Aumann and Maschler (1995), KG study when and how the sender can improve his own expected payoff through "persuasion": choosing an experiment and committing to revealing its signal.

My analysis and motivation differs from that of KG in several ways. Rather than studying when the sender might benefit from persuasion, this paper investigates how observed choices can be used to test whether the sender and receiver conform to the KG framework. In particular, the representation characterizes what it means for the receiver to be a

[6]In contrast, the cheap talk literature assumes the sender has no commitment power; see, for example, Crawford and Sobel (1982).

Bayesian information processor and the sender a sophisticated planner. The characterization is expressed in terms of the choices agents actually make in the KG framework—the sender chooses information, and the receiver chooses among risky alternatives. I also show how observed choices can be used to identify and compare the parameters (beliefs and utilities) of the agents. While KG take as given a fixed set of actions and consider a sender who is free to choose his most-preferred information structure, my analysis involves a rich set of choice data: for each menu of acts, the sender's full ranking of information structures is observed. The full ranking is needed to characterize the agents and identify parameters. Finally, while KG assume common priors, my framework permits the sender and receiver to hold different priors.[7]

Dynamic inconsistency and related conceptual issues have been studied in a variety of settings by several authors. For example, Epstein and Le Breton (1993) show that if an individual is dynamically consistent, then his beliefs can be represented by a probability distribution. Karni and Schmeidler (1991) show that preferences over conditional lotteries satisfying standard assumptions are dynamically consistent if and only if they are expected utility preferences. Similarly, Border and Segal (1994) show that upon observing low-probability events, conditional preferences are well-approximated by expected utility preferences provided the individual is dynamically consistent and has differentiable ex-ante preferences. Grant, Kajii, and Polak (2000) examine when a dynamically consistent individual with non-expected utility preferences prefers more information to less. These papers take underlying preferences (in some cases, corresponding utility representations) as given and examine implications of dynamic consistency or inconsistency. In contrast, the main primitive of my model is an individual's ranking of information structures themselves, from which preferences and beliefs can be identified and dynamic consistency tested.

Behavioral economists have developed models where information suppression or self-signaling can be used to regulate behavior. Carrillo and Mariotti (2000) show that, in a model of personal equilibrium, time-inconsistent agents may benefit from acquiring less information. Benabou and Tirole (2002, 2006) study equilibrium models where players rationally limit the information available to future selves. In the persuasion literature, Lipnowski and Mathevet (2016) examine how a benevolent principal should disclose information to agents who are susceptible to temptation, reference-dependence, or other behavioral considerations. In a similar spirit, the behavioral interpretation of my model provides a general analysis of the incentives for information acquisition for individuals lacking time-consistent preferences or prior beliefs.

[7] Alonso and Câmara (2016) extend the KG framework to allow heterogeneous priors and find that (generically) the sender benefits from persuasion under heterogeneous priors.

Azrieli and Lehrer (2008) consider preferences over information structures and provide necessary and sufficient conditions for such a preference to be represented by expected utility in some decision problem.[8] In their representation, a utility index and a menu of actions are deduced from the preference for information (the prior is taken as given), but cannot be uniquely pinned down. Azrieli and Lehrer (2008) note that their axioms can be modified to allow an endogenous prior but that it, too, cannot be uniquely identified. My model resolves these identification issues by examining preferences for information in all (exogenously specified) menus—even with time-inconsistent priors or utilities, all parameters can be uniquely identified from this richer collection of preferences.

Several authors have studied Bayesian updating from a decision-theoretic perspective. Ghirardato (2002) develops a representation using conditional preferences over acts; that is, families of preferences indexed by events, with the interpretation that the event represents an observed signal. Karni (2007) uses a similar family of conditional preferences defined over conditional acts. The extra structure of conditional acts permits both prior beliefs and state-dependent utilities to be identified, in addition to testing Bayesian updating of partitional information. Wang (2003) axiomatizes Bayes' rule and some of its extensions in a setting with conditional preferences over (infinite-horizon) consumption-information profiles; preferences are conditioned on sequences of previously realized events. My representation characterizes Bayesian updating using signal-contingent preferences over standard Anscombe-Aumann acts. Importantly, the set of signals is richer than the state space over which acts are defined, enabling a simple and intuitive characterization.

Finally, Lu (2016) shows how random choice data reveals an individual's information, provided the individual is a Bayesian subjective expected utility maximizer. Decision-theoretic models of rational inattention[9] also use standard choice primitives to make inferences about an individual's preferences, beliefs, and information processing ability. I take the opposite approach and use an individual's informational choice to make inferences about his underlying tastes and beliefs.

[8] See also Gilboa and Lehrer (1991), who study a similar problem for the case of partitional information structures.

[9] See Denti, Mihm, de Oliveira, and Ozbek (2016), Ellis (2018), and Caplin and Dean (2015)

2 Framework and Notation

2.1 Outcomes, lotteries, acts

Let X denote a finite set of $N \geq 2$ *outcomes*. Elements of X are typically denoted x, y, while elements of ΔX (*lotteries*) are denoted p, q.[10] A lottery p assigns probability $p(x)$ to outcome x.

A *utility index* is a function $u : X \to \mathbb{R}$. If $p \in \Delta X$ and u is a utility index, let $u(p) := \sum_{x \in X} u(x)p(x)$ denote the expected utility of p. A utility index u' is a positive affine transformation of u if there are real numbers $A > 0$ and B such that $u'(x) = Au(x) + B$ for all $x \in X$. The notation $u' \approx u$ indicates that u' is a positive affine transformation of u.

There is a finite, exogenous state space $\Omega = \{1, \ldots, W\}$ where $W \geq 2$ denotes the number of states. Arbitrary states are typically denoted ω, ω', while members of $\Delta\Omega$ (probability distributions over Ω) are denoted μ or ν.

As a notational convention, subscripts denote states. For example, a distribution $\mu \in \Delta\Omega$ may be expressed as $\mu = (\mu_\omega)_{\omega \in \Omega}$, where μ_ω is the probability assigned to state ω.

A function $f : \Omega \to \Delta X$ is an (Anscombe-Aumann) *act*. Let F denote the set of all acts. Acts are typically denoted f, g, h, and may be written as profiles: $f = (f_\omega)_{\omega \in \Omega}$, where $f_\omega \in \Delta X$. The set F is equipped with the standard mixing operation: if $f, g \in F$ and $\alpha \in [0, 1]$, then $\alpha f + (1 - \alpha)g := (\alpha f_\omega + (1 - \alpha)g_\omega)_{\omega \in \Omega}$.

A *menu* is a finite, nonempty set of acts. Menus are typically denoted A, B. Let $\mathcal{A}$ denote the set of all menus.

2.2 Blackwell Experiments

Definition 1 (Blackwell Experiment)**.** A matrix σ with entries in $[0, 1]$ is a (finite) *Blackwell experiment* if it has exactly W rows, no columns consisting only of zeros and, for each row, the sum of entries is exactly one. Let $\mathcal{E}$ denote the set of all Blackwell experiments.

Implicitly, each column of σ represents a signal that might be generated. Thus, each row represents a state-contingent probability distribution over a finite set of signals. The assumption that each column contains at least one nonzero entry eliminates signals that have zero probability of occurrence in each state. Note that entries in any given column are not required to sum to one.

[10] For any finite set Y, ΔY denotes the standard probability simplex over Y, equipped with the usual convex mixture operation.

It will be convenient to express experiments in terms of their columns. Let

$$S := \{s = (s_\omega)_{\omega \in \Omega} \in [0,1]^\Omega : \exists \omega \text{ such that } s_\omega \neq 0\} \tag{2}$$

Elements of S are called *signals*. Clearly, every column of an experiment σ corresponds to a signal s where s_ω is the entry for the column in row ω.

The statement '$s \in \sigma$' means s is a column of σ. Note that an experiment may have duplicate columns. When quantifying over signals in an experiment, different columns of σ are distinguished even if they are duplicates. For example, the requirement that each row in σ has entries summing to one may be expressed as '$\forall \omega$, $\sum_{s\in\sigma} s_\omega = 1$' because the summation notation implicitly distinguishes between duplicate columns of σ. Similarly, statements like '$\forall s \in \sigma$, $y^s \in Y$' associate (potentially) different members of Y to different columns of σ, even if those columns are duplicates.

For each σ and $\alpha \in (0,1)$, let $\alpha\sigma$ denote the matrix formed by multiplying each entry of σ by α. If $\sigma, \sigma' \in \mathcal{E}$ and $\alpha \in (0,1)$, then $\alpha\sigma \cup (1-\alpha)\sigma'$ denotes the matrix consisting of the columns of $\alpha\sigma$ together with the columns of $(1-\alpha)\sigma'$. It is easy to verify that this mixture yields a well-defined experiment.[11] If $\alpha \in \{0,1\}$, then $\alpha\sigma \cup (1-\alpha)\sigma'$ refers either to σ (when $\alpha = 1$) or to σ' (when $\alpha = 0$).

2.3 Primitives

I take as primitive two sets of choice data:

(1) For each menu $A \in \mathcal{A}$, a preference $\succsim^A$ over $\mathcal{E}$

(2) For each signal $s \in S$, a choice correspondence c^s such that, for each menu A, $c^s(A)$ is a nonempty subset of A.

The family $(\succsim^A)_{A\in\mathcal{A}}$ captures the sender's preferences for information. In particular, $\sigma \succsim^A \sigma'$ means the sender expects a higher average payoff from σ than from σ', given that the receiver observes a signal generated from the chosen experiment before choosing from A. Hence, the sender does not choose from A and cannot restrict the choices available to the receiver; he can only influence the receiver's choice by controlling the available information.

The receiver's choices are captured by the collection $(c^s)_{s\in S}$. For each signal s and menu A, $c^s(A) \subseteq A$ consists of the acts from A that are chosen by the receiver after observing signal s. In practice, the receiver's choice is conditioned on a pair (σ, s) where $s \in \sigma$; that

[11] Note that this operation is not commutative. Specifically, $\alpha\sigma \cup (1-\alpha)\sigma'$ means the matrix $(1-\alpha)\sigma'$ is appended to the right of matrix $\alpha\sigma$. So, typically, $\alpha\sigma \cup (1-\alpha)\sigma' \neq (1-\alpha)\sigma' \cup \alpha\sigma$.

is, the receiver must know both σ as well as the signal s generated by σ. However, for a Bayesian information processor, only the entries of s matter. To minimize notation, I condition choices on signals s instead of pairs (σ, s).[12]

3 The Representation

The objective is to represent the sender's preferences for information in terms of subjective expected utility and the receiver's signal-contingent behavior as subjective expected utility with Bayesian updating. Throughout, I will often refer to the two decision makers as DM1 (the sender) and DM2 (the receiver).

First, consider the receiver. In the representation, the receiver has a full-support prior μ and a utility index u.

Definition 2 (Bayesian Representation)**.** A pair (μ, u) is a *Bayesian Representation* for $(c^s)_{s\in S}$ if $\mu \in \Delta\Omega$ has full support, $u : X \to \mathbb{R}$ is a non-constant utility index and, for all $s \in S$ and $A \in \mathcal{A}$,

$$c^s(A) = \left\{ f \in A \ : \ \forall g \in A, \ \sum_{\omega} u(f_\omega)\mu^s_\omega \geq \sum_{\omega} u(g_\omega)\mu^s_\omega \right\} \tag{3}$$

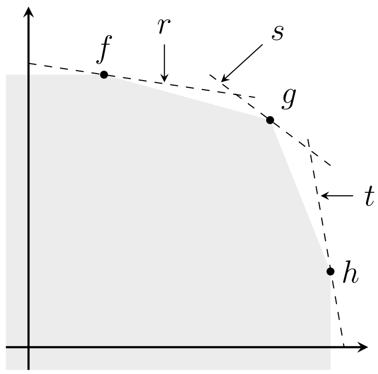
where the posteriors μ^s satisfy Bayes rule:

$$\forall \omega \in \Omega, \ \mu^s_\omega = \frac{\mu_\omega s_\omega}{\sum_{\omega' \in \Omega} \mu_{\omega'} s_{\omega'}} \tag{4}$$

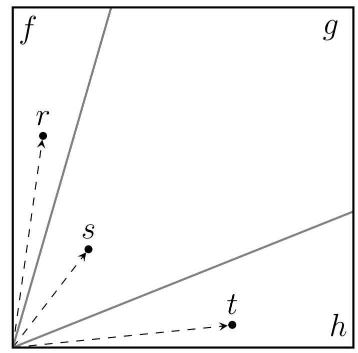
In a Bayesian representation, each choice correspondence c^s is rationalized by expected utility maximization with prior μ, utility index u, and Bayesian updating. That is, upon observing signal s, DM2 updates his prior μ to the Bayesian posterior μ^s given by (4). Then, he chooses $f \in A$ if and only if f maximizes expected utility under beliefs μ^s and utility index u. Figure 1 provides geometric representations of this behavior.

Next, consider the sender. The key to understanding his preference for information is to examine how, from his perspective, an experiment σ transforms into an Anscombe-Aumann act. Fix a state ω. Since σ specifies a distribution over signals for state ω, and since the receiver's choice from A only depends on the realized signal, the distribution of signals becomes a distribution of choices (acts) from A. Evaluating the acts at state ω turns the distribution of acts into a distribution of lotteries, which the sender reduces to a single

[12]If choices were conditioned on pairs (σ, s), the following additional axiom would be required: for all σ, σ' with $s \in \sigma$ and $s \in \sigma'$, $c^{(\sigma,s)} = c^{(\sigma',s)}$.



(a) Utility Space



(b) Signal space

Figure 1: Geometric representations of DM2's behavior when $|\Omega| = 2$. DM2 prefers f over g at signal s if and only if $\sum_\omega u(f_\omega)s_\omega\mu_\omega \geq \sum_\omega u(g_\omega)s_\omega\mu_\omega$. Thus, in utility space, acts f correspond to points $(\mu_1 u(f_1), \mu_2 u(f_2))$, and choices at signals s are determined by the ratio s_1/s_2. Consequently, DM2's choices from $A = \{f, g, h\}$ partition S into convex cones. The arrows pointing to signals in (b) are perpendicular to the corresponding lines in (a).

lottery. Repeating this procedure for each state yields a lottery for each state and, hence, an Anscombe-Aumann act called an *induced act*. The next definition formalizes this process.

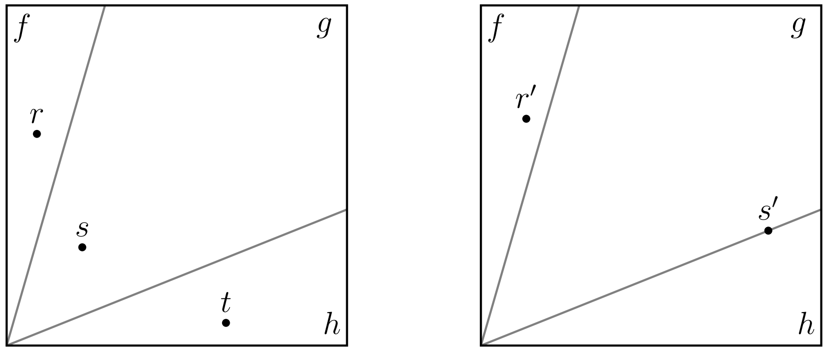
Definition 3 (Induced Acts)**.** The set of *induced acts* for experiment σ at menu A is given by

$$F^A(\sigma) := \left\{ \left(\sum_{s\in\sigma} s_\omega f^s_\omega \right)_{\omega\in\Omega} : \ \forall s \in \sigma, \ f^s \in \Delta c^s(A) \right\} \tag{5}$$

For each menu A, let $\mathcal{E}^*(A)$ denote the set of experiments such that $F^A(\sigma)$ is a singleton. If $\sigma \in \mathcal{E}^*(A)$, then $F^A_\omega(\sigma) := f_\omega$, where $F^A(\sigma) = f \in F$.

Definition 3 generalizes the idea above to include the possibility of ties—signals that make DM2 indifferent between two or more acts. If a signal s results in a tie, then acts $f^s \in \Delta c^s(A)$ represent randomizations over the acts that DM2 might choose from A at s. Since s occurs with probability s_ω in state ω, this yields a lottery $\sum_{s\in\sigma} s_\omega f^s_\omega \in \Delta X$ for state ω. Repeating this procedure for each state ω yields an Anscombe-Aumann act, and letting f^s vary across all members of $\Delta c^s(A)$ generates a set of acts. This set, denoted $F^A(\sigma)$, encapsulates the full range of possibilities for DM2's behavior given that second period choices must be consistent with the correspondences c^s. Figure 2 illustrates the procedure.

Note that $F^A(\sigma)$ is defined using only the receiver's choice behavior, and that no assumptions (other than non-emptiness) about the choice correspondences are required to compute $F^A(\sigma)$. Since an experiment σ consists of finitely many signals, $F^A(\sigma)$ references only finitely many observations of DM2's choices. Hence, testable properties of DM1's preferences can be expressed in terms of induced acts.



(a) $\sigma = [r, s, t]$ (b) $\sigma' = [r', s']$

Figure 2: Induced acts. In (a), the induced act is $F^A(\sigma) = (r_1 f_1 + s_1 g_1 + t_1 h_1, r_2 f_2 + s_2 g_2 + t_2 h_2)$. In (b), signal s' results in a tie between g and h, and therefore $F^A(\sigma') = \{(r'_1 f_1 + s'_1[\alpha g_1 + (1-\alpha)h_1], r'_2 f_2 + s'_2[\alpha g_2 + (1-\alpha)h_2]) : \alpha \in [0,1]\}$.

When $F^A(\sigma)$ contains exactly one act, the sender evaluates the act using subjective expected utility. If $F^A(\sigma)$ contains more than one act, however, then standard expected utility cannot unambiguously assign a value to this set. The representation permits a high degree of freedom regarding the evaluation of such sets; only basic requirements of linearity and consistency, defined next, are imposed.

Definition 4. A family of functions $(V^A : \mathcal{E} \to \mathbb{R})_{A \in \mathcal{A}}$ is:

1. A *representation* of $(\succsim^A)_{A \in \mathcal{A}}$ if $V^A(\sigma) \geq V^A(\sigma') \Leftrightarrow \sigma \succsim^A \sigma'$
2. *Linear* if $V^A(\alpha\sigma \cup (1-\alpha)\sigma') = \alpha V^A(\sigma) + (1-\alpha) V^A(\sigma')$
3. *Consistent* if $V^A(\sigma) \geq V^A(\sigma') \Leftrightarrow V^B(\hat{\sigma}) \geq V^B(\hat{\sigma}')$ whenever $F^A(\sigma) = F^B(\hat{\sigma})$ and $F^A(\sigma') = F^B(\hat{\sigma}')$.

In other words, linear functions are separable with respect to the mixture operation on experiments, and a consistent family of representations derives from some common ranking of convex sets of acts.

Definition 5 (Value of Information Representation)**.** A family $(V^A : \mathcal{E} \to \mathbb{R})_{A \in \mathcal{A}}$ of consistent, linear representations and a pair (ν, v) constitute a *Value of Information Representation* for $(\succsim^A)_{A \in \mathcal{A}}$ if $\nu \in \Delta\Omega$ has full support, $v : X \to \mathbb{R}$ is a non-constant utility index and, for each menu A and all $\sigma \in \mathcal{E}^*(A)$,

$$V^A(\sigma) = \sum_{\omega} v(F^A_\omega(\sigma))\nu_\omega \tag{6}$$

Definition 5 requires that, whenever $F^A(\sigma)$ consists of a single act, DM1 computes the expected utility of that act. For experiments involving ties (signals making DM2 indifferent between two or more acts), the only constraints on V^A are linearity and consistency. Thus, the representation is fairly silent regarding the sender's attitude toward ties. DM1 may adhere to sender-preferred tie-breaking, as is typically assumed in the literature, but such a rule is not required.[13] Instead of best-case beliefs, he might hold worst-case beliefs, or perhaps receive bonus utility or disutility from the presence of ties—these (and many other) rules satisfy linearity and consistency. The advantage of Definition 5 is that it does not impose a specific, arbitrary tie-breaking rule and, hence, does not require an additional axiom characterizing such a rule.[14]

3.1 Characterization of DM1

In this section I focus on DM1. I present six axioms employing both first-period preferences $\succsim^A$ and second-period choices c^s, and show that if DM2 has a Bayesian representation, then these axioms are necessary and sufficient for DM1 to have a Value of Information representation with standard uniqueness properties. In section 3.2 I present five axioms on second-period choices c^s and show that they are necessary and sufficient for DM2 to have a Bayesian representation, also with standard uniqueness properties. A combined representation theorem characterizing both decision makers follows immediately.

The first three axioms for DM1 are standard vNM axioms, adapted to operate on Blackwell experiments and their mixtures. The Independence and Continuity axioms hold by the linearity requirement for Value of Information representations and the fact that (i) $F^A(\sigma)$ is convex for all A and σ, and (ii) $F^A(\alpha\sigma \cup (1-\alpha)\sigma') = \alpha F^A(\sigma) + (1-\alpha)F^A(\sigma')$ whenever DM2 has a Bayesian representation[15]; see the appendix for proofs.

Axiom A1 (Rationality). Each $\succsim^A$ is complete and transitive.

Axiom A2 (Independence). If $\sigma \succ^A \sigma'$ and $\alpha \in (0,1)$, then $\alpha\sigma \cup (1-\alpha)\sigma'' \succ^A \alpha\sigma' \cup (1-\alpha)\sigma''$ for all σ''.

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[13]This means the sender ranks experiments as if, at every signal s resulting in a tie, the receiver selects an act in $c^s(A)$ that is most-preferred by the sender. Formally, each $\succsim^A$ is represented by the function $\overline{V}^A(\sigma) := \max_{f \in F^A(\sigma)} \sum_\omega v(f_\omega)\nu_\omega$.

[14]To expand on this point, other models achieve best-case tie-breaking rules by imposing an appropriate upper semi-continuity axiom. Such axioms are not testable. In the present framework, formulating an appropriate continuity assumption introduces many complications because—unlike preferences over acts or menus—preferences $\succsim^A$ exhibit severe discontinuities (in both σ and A) even under sender-preferred tie-breaking. Focusing on the more general class of preferences given by Definition 5 circumvents these problems and makes tie-breaking attitude a subjective characteristic of the individual: his attitude is revealed by his preferences.

[15]For convex sets $X, Y \subseteq F$ and $\alpha \in [0,1]$, let $\alpha X + (1-\alpha)Y := \{\alpha f + (1-\alpha)g : f \in X, g \in Y\}$.

Axiom A3 (Continuity). If $\sigma \succ^A \sigma' \succ^A \sigma''$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha\sigma \cup (1-\alpha)\sigma'' \succ^A \sigma' \succ^A \beta\sigma \cup (1-\beta)\sigma''$.

Although these axioms are familiar, the Mixture Space Theorem (Herstein and Milnor, 1953) does not apply because the set $\mathcal{E}$ with the given mixture operation does not qualify as a mixture space. The final axiom, Consistency, will help circumvent this problem.

For $p \in \Delta X$, $h \in F$, and $\omega \in \Omega$, let $p[\omega]h$ denote the act formed by taking h and replacing h_ω with p. The next axiom is analogous to the State Independence axiom in the Anscombe-Aumann model, once again adapted to operate on experiments.[16] The version presented here rules out null states, so that DM1's prior ν will have full support.

Axiom A4 (State Independence). Suppose $F^A(\sigma) = p[\omega]h$ and $F^A(\sigma') = q[\omega]h$ while $F^A(\hat{\sigma}) = p[\omega']\hat{h}$ and $F^A(\hat{\sigma}') = q[\omega']\hat{h}$. Then $\sigma \succsim^A \sigma'$ implies $\hat{\sigma} \succsim^A \hat{\sigma}'$.

Recall that for each menu A, $\mathcal{E}^*(A) \subseteq \mathcal{E}$ denotes the set of experiments σ such that $F^A(\sigma)$ is single-valued. The following axiom is needed to disentangle DM1's beliefs and utilities.

Axiom A5 (Non-Degeneracy). There is a menu A and experiments $\sigma, \sigma' \in \mathcal{E}^*(A)$ such that $\sigma \succ^A \sigma'$.

Finally, Axiom A6 states that DM1's ranking of two experiments is determined by their associated sets of induced acts. This is the only axiom asserting any relationships between different orderings $\succsim^A$ and $\succsim^B$.

Axiom A6 (Consistency). If $F^A(\sigma) = F^B(\hat{\sigma})$ and $F^A(\sigma') = F^B(\hat{\sigma}')$, then $\sigma \succsim^A \sigma'$ implies $\hat{\sigma} \succsim^B \hat{\sigma}'$.

Consistency serves two purposes. First, suppose $F^A(\sigma) = F^A(\sigma')$. In this case, Consistency forces $\sigma \sim^A \sigma'$. Thus, a given preference $\succsim^A$ can be transformed into a ranking of convex sets of acts. As demonstrated in the appendix, this means each $\succsim^A$ can be embedded in a mixture space and that axioms A1–A3 translate into the standard vNM axioms via the embedding. Hence, a linear representation can be derived for each $\succsim^A$ even though $\mathcal{E}$, equipped with the mixing operation defined above, does not qualify as a mixture space.

[16] The standard axiom says: if ω, ω' are non-null and $p[\omega]h$ is weakly preferred over $q[\omega]h$, then $p[\omega']\hat{h}$ is weakly preferred over $q[\omega']\hat{h}$ for all $\hat{h}$.

Second, suppose $F^A(\sigma) = \{f\} = F^B(\hat{\sigma})$ and $F^A(\sigma') = \{g\} = F^B(\hat{\sigma}')$. Then, by Consistency, DM1 ranks $\sigma \succsim^A \sigma'$ if and only if $\hat{\sigma} \succsim^B \hat{\sigma}'$. Thus, it is as if these rankings derive from a single ordering of f and g. This is needed for DM1 to hold a utility index and prior beliefs that do not depend on the menu under consideration. If $\{f\}$ and $\{g\}$ are replaced with arbitrary (non-singleton) sets of acts, then, in a similar spirit, the axiom requires DM1's ranking of the sets to be independent of the menu under consideration.

Theorem 1. *Suppose $(c^s)_{s\in S}$ has a Bayesian representation. The collection $(\succsim^A)_{A\in\mathcal{A}}$ satisfies Axioms A1–A6 if and only if it has a Value of Information Representation. Moreover, ν is unique and, for each A, V^A (hence, v) is unique up to positive affine transformation.*

Although all but one of the axioms for DM1 are adaptations of the Anscombe-Aumann axioms to this setting, Theorem 1 is not a direct corollary of the Anscombe-Aumann theorem. There are two obstacles. First, it is not obvious that variation in σ induces enough variation in DM2's choices to establish existence of an expected utility representation for DM1, or uniqueness of ν and v provided a representation exists. In particular, consider the set $F^A := \{F^A(\sigma) : \sigma \in \mathcal{E}^*(A)\}$ of induced acts under menu A. The restriction of $\succsim^A$ to $\mathcal{E}^*(A)$ transforms into a ranking over F^A. Although F^A is convex, it is typically a proper subset of F. Therefore, depending on μ, u, and the menu A under consideration, the axioms may not be active on a sufficiently rich domain of induced acts to establish existence or uniqueness of an expected utility representation for DM1. A key step of the proof constructs a menu A^* from which existence is established and candidates for ν and v can be uniquely identified.

Second, it is also not obvious that the Consistency axiom is strong enough to ensure unique (menu-independent) beliefs and utilities can be obtained for DM1. If two menus have disjoint sets of induced acts, then Consistency seemingly has no bite and DM1 could hold different beliefs and/or utilities in those menus. The main challenge of the proof is to show that any two menus can be connected by a finite sequence of menus with significantly overlapping sets of induced acts along the way, thus ensuring uniqueness. In section 4, I sketch the steps needed to prove Theorem 1; for a complete proof, please see the appendix.

3.2 Characterization of DM2

While the axiomatic characterization of DM1 requires both the ex-ante preferences $(\succsim^A)_{A\in\mathcal{A}}$ of DM1 and ex-post choice correspondences $(c^s)_{s\in S}$ of DM2, the characterization of DM2's Bayesian Representation only requires the correspondences $(c^s)_{s\in S}$.

Axiom B1 (Rationality). Each choice function c^s satisfies WARP.

Since the model restricts attention to finite menus, Axiom B1 implies that for each s there is a complete and transitive relation $\succsim^s$ rationalizing c^s. Specifically, $f \succsim^s g$ if and only if $f \in c^s(\{f, g\})$.

Axiom B2 (Non-Degeneracy). For each s, there are acts f, g such that $f \succ^s g$.

This is a standard axiom in the Anscombe-Aumann model. It says that each relation $\succsim^s$ does not simply assign indifference among all acts, and is needed to disentangle preferences and beliefs.

Next, endow F with the standard Euclidean topology and S with the topology of real projective space.[17] Let $F \times S$ employ the corresponding product topology.

Axiom B3 (Continuity). For each f, the sets $\{(g, s) \in F \times S : f \succsim^s g\}$ and $\{(g, s) \in F \times S : g \succsim^s f\}$ are closed.

This axiom expresses two forms of continuity. First, holding s constant, it says that $\succsim^s$ satisfies the usual continuity: contour sets of $\succsim^s$ are closed. Second, holding f and g constant, it says that if $f \succ^s g$, then it is possible to perturb s while maintaining strict preference for f over g. This holds because the Bayesian posterior μ^s varies continuously with s in the given topology.

If $E \subseteq \Omega$ and $f, h \in F$, let fEh denote the act g such that $g_\omega = f_\omega$ for $\omega \in E$ and $g_\omega = h_\omega$ otherwise. Similarly, if $s, t \in S$, let sEt denote the profile r such that $r_\omega = s_\omega$ for $\omega \in E$ and $r_\omega = t_\omega$ otherwise. Note that r may not be a well-defined signal (to qualify as a signal, at least one entry of r must be nonzero).

Axiom B4 (State Independence). Suppose $f = p[\omega]h$ and $g = q[\omega]h$ while $f' = p[\omega']h'$ and $g' = q[\omega']h'$. If $s_\omega, s'_{\omega'} > 0$ and $f \succsim^s g$, then $f' \succsim^{s'} g'$.

This is a slight modification of the standard State Independence axiom used in the Anscombe-Aumann model. It rules out null states for DM1's prior (so the prior μ will have full support) and ensures the existence of a common ranking over lotteries independently of the state and independently of the preference $\succsim^s$ under consideration.

Axiom B5 (Bayesian Independence). If $f \succ^s g$, $\alpha \in (0, 1)$ and $t = sE(\alpha s)$, then

[17] This is the quotient topology of the standard Euclidean topology on $[0,1]^\Omega \backslash 0$ with respect to the equivalence relation $s \sim \lambda s$ for all $\lambda > 0$.

$(\alpha f + (1-\alpha)h)Ef \succ^t (\alpha g + (1-\alpha)h)Eg$ for all h.

To understand this axiom, suppose $f \succ^s g$. When comparing $(\alpha f + (1-\alpha)h)Ef$ and $(\alpha g + (1-\alpha)h)Eg$, an expected utility maximizer "cancels" the $(1-\alpha)h$ from both acts to yield a comparison between "$(\alpha f)Ef$" and "$(\alpha g)Eg$". Effectively, this scales down the utilities from acts f and g by a factor of α on event E, making event $E^c := \Omega\backslash E$ more attractive. Bayesian Independence says that scaling down the likelihood of states in E^c by the same factor α compensates for this change: the ranking is preserved under signal $t = sE(\alpha s)$. Thus, Bayesian Independence expresses an equivalence between scaling utilities and scaling state likelihoods. Note that when $E = \Omega$, Bayesian Independence reduces to the standard independence axiom for $\succsim^s$.

Theorem 2. *The collection $(c^s)_{s\in S}$ satisfies axioms B1–B5 if and only if it has a Bayesian representation (μ, u). Furthermore, μ is unique and u is unique up to positive affine transformation.*

The proof of Theorem 2 is fairly straightforward. First, observe that each $\succsim^s$ satisfies the Anscombe-Aumann axioms. In particular, Bayesian Independence implies the standard Independence axiom, and the standard Continuity axiom is implied by axiom B3. Hence, $\succsim^s$ has an expected utility representation with parameters (μ^s, u^s), where μ^s is a prior and u^s a utility index. Axiom B4 implies that $\succsim^s$ and $\succsim^{s'}$ have the same ranking over constant acts (lotteries), so it is without loss of generality to assume $u^s = u$ for all s.

The only remaining task is to ensure that the priors μ^s are the correct Bayesian posteriors given signal s for some prior μ. The natural candidate for μ is μ^e, where $e_\omega = 1$ for all ω, because the signal e provides no new information. Essentially, the Bayesian Independence axiom ensures that the probability ratios satisfy $\frac{\mu^s_\omega}{\mu^s_{\omega'}} = \frac{s_\omega \mu^e_\omega}{s_{\omega'} \mu^e_{\omega'}}$, as prescribed by Bayes' rule. For full detail, please see the appendix.

The following result is an immediate consequence of Theorems 1 and 2.

Theorem 3. *The collections $(\succsim^A)_{A\in\mathcal{A}}$ and $(c^s)_{s\in S}$ satisfy Axioms A1–A6 and B1–B5 if and only if the sender has a Value of Information representation and the receiver a Bayesian representation. The associated priors ν, μ are unique, and the utility indices v, u are unique up to positive affine transformation.*

Proof. First apply Theorem 2 to establish a Bayesian representation (μ, u) for DM2 with the desired uniqueness properties. Then apply Theorem 1 to establish a Value of Information representation for DM1, also with the desired uniqueness properties. □

4 Outline of the Proof

In this section, I outline the key steps needed to prove Theorem 1. For simplicity, I restrict attention to the case of two states and three outcomes. For a complete proof, please see the appendix.

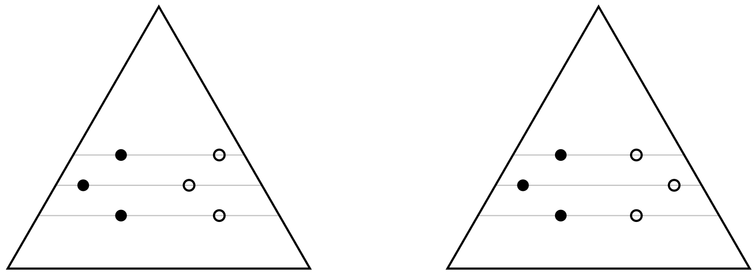
The first step is to establish a linear representation V^A for each $\succsim^A$. By the Consistency axiom, $\succsim^A$ translates into an ordering on $\mathcal{M}^A := \{F^A(\sigma) : \sigma \in \mathcal{E}\}$, a family of convex sets of acts. Recall that $F^A(\alpha\sigma \cup (1-\alpha)\sigma') = \alpha F^A(\sigma) + (1-\alpha)F^A(\sigma')$ when DM2 has a Bayesian representation. It follows that the set $\mathcal{M}^A$ (equipped with the standard mixing operation for convex sets) is a mixture space and that (by Consistency) Axioms A1–A3 translate into the von Neumann-Morgenstern axioms on $\mathcal{M}^A$. Thus, the mixture space theorem gives a linear representation on $\mathcal{M}^A$, which (by Consistency) maps back to a linear V^A on $\mathcal{E}$.

The second step constructs a menu A such that the set of induced acts is rich enough to identify candidates for ν and v. The key is to find a full-dimensional set $L \subseteq \Delta X$ and lotteries p, q such that $L \times p$ and $q \times L$ are subsets of $F^A := \{F^A(\sigma) : \sigma \in \mathcal{E}^*(A)\} \subseteq F$, the set of all induced acts (not involving ties) for menu A. This way, the State Independence axiom implies there is a unique von Neumann-Morgenstern utility index, v, that applies in all states, ensuring a separation of beliefs and utilities. The construction of A and verification of its properties is contained in the appendix.

The third step establishes that the parameters (ν, v) are not menu-dependent. More precisely, the proof shows there is a unique, non-degenerate, linear preference $\succsim$ on F from which every $\succsim^A$ derives: if $F^A(\sigma) = f$ and $F^A(\sigma') = g$, then $\sigma \succsim^A \sigma'$ if and only if $f \succsim g$. Thus, when there are no ties, preferences are determined by $\succsim$ and a unique pair (ν, v) exists.

Consistency (Axiom A6) plays a key role here. Clearly, $\succsim^A$ is associated with a linear representation on F^A. If F^A has full dimension, then a linear preference on F^A has a unique linear extension, $\succsim$, to all of F. Therefore, if $\dim F^B \leq \dim(F^A \cap F^B)$, the linear relation on F^B (corresponding to $\succsim^B$) agrees with $\succsim$. In this sense, B *inherits* a representation from A. The proof, therefore, establishes two facts: (i) every B inherits a representation from some A where F^A has full dimension; and (ii) if F^A and F^B have full dimension, then $\succsim^A$ and $\succsim^B$ derive from the same linear ordering $\succsim$ on F.

Part (ii) is the crucial step. If F^A and F^B each have full dimension, but $F^A \cap F^B = \emptyset$, then Consistency seemingly has no bite and $\succsim^A$ and $\succsim^B$ may require different choices of ν or v (if expected utility representations even exist for these preferences). The core of the argument establishes that for any such pair of menus, there exists a finite sequence of menus $A = A^0, A^1, \ldots, A^n = B$ such that, for all $0 \leq i < n$, $F^{A^i} \cap F^{A^{i+1}}$ has full dimension. Then, Consistency forces each $\succsim^{A^i}$ to derive from the same $\succsim$ on F.



(a) Oriented (b) Not Oriented

Figure 3: Orientedness when $|X| = 3$. The lotteries $A_\omega = \{f_\omega, g_\omega, h_\omega\}$ are solid dots and the lotteries $B_\omega = \{f'_\omega, g'_\omega, h'_\omega\}$ are circles. In this case, $\lambda = 0$ (the lines are indifference curves for u). The configuration in (b) is not oriented because the affine path from A_ω to B_ω (traversing along the lines) yields a collinear set of lotteries at $\alpha = 1/2$.

An important special case is when A and B are *oriented translations* of each other. With two states and three outcomes, this means we can write $A = \{f, g, h\}$ and $B = \{f', g', h'\}$ with the following properties. First, there exists $\lambda \in \mathbb{R}^2$ such that, in utility space, $f' = f + \lambda$, $g' = g + \lambda$, and $h' = h + \lambda$. That is, $(u(f'_1)\mu_1, u(f'_2)\mu_2) = (u(f_1)\mu_1, u(f_2)\mu_2) + \lambda$ (and similar formulas for g' and h'). This is the *translation* property. Second, for every ω and $\alpha \in [0, 1]$, the set $A^\alpha_\omega := \{\alpha f'_\omega + (1-\alpha)f_\omega, \alpha g'_\omega + (1-\alpha)g_\omega, \alpha h'_\omega + (1-\alpha)h_\omega\}$ is affinely independent. This is the *orientedness* property, and it ensures that the convex hull of A^α_ω has full dimension in ΔX for all α and ω (see Figure 3). Third, f, g, h are chosen in such a way that for each of the three acts, there is a signal s such that DM2 strictly prefers that act after observing s.

When A and B are oriented translations, there is a smooth (linear) path from A to B where the set of induced acts has full dimension along the way. Specifically, the path is given by $A^\alpha := \{\alpha f' + (1-\alpha)f, \alpha g' + (1-\alpha)g, \alpha h' + (1-\alpha)h\}$ for $\alpha \in [0, 1]$. Smoothness yields the desired sequence of menus $A = A^0, A^1, \ldots, A^n = B$ by taking an appropriate finite set of points α. The sets of induced acts for adjacent menus have full-dimensional intersections, forcing $\succsim^A$ and $\succsim^B$ to derive from the same preference on F and, hence, employ the same prior and utility function.

It turns out that, after a series of transformations, one can restrict attention to menus A and B that are oriented translations of one another. The construction involves several steps and is relegated to the appendix.

5 Identification

Theorem 3 characterizes the behavior of DM1 and DM2 in terms of the primitives $(\succsim^A)_{A \in \mathcal{A}}$ and $(c^s)_{s \in S}$ and shows that ν, μ, v, and u can be identified from those primitives. The goal of this section is to establish a stronger identification result: all four parameters can be identified using only DM1's preferences $(\succsim^A)_{A\in\mathcal{A}}$.

The analysis is carried out by considering two individuals, DM and $\dot{\text{D}}$M, each decomposed into two decision makers: DM = (DM1,DM2), and $\dot{\text{D}}$M = ($\dot{\text{D}}$M1,$\dot{\text{D}}$M2). DM and $\dot{\text{D}}$M are characterized by data $(\succsim^A, c^s)_{A\in\mathcal{A},s\in S}$ and $(\dot{\succsim}^A, \dot{c}^s)_{A\in\mathcal{A},s\in S}$, respectively, satisfying axioms A1–A6 and B1–B5. Thus, their behavior is captured by parameters $((\nu, v),(\mu, u))$ for DM and $((\dot{\nu}, \dot{v}),(\dot{\mu}, \dot{u}))$ for $\dot{\text{D}}$M. Note that this section only concerns identification of the parameters. Hence, the results are meaningful for an outside observer who, rather than testing the axioms, is willing to assume they are satisfied and wishes to identify the priors and utilities of the sender and receiver.

For convenience, I assume sender-preferred tie-breaking for both DM and $\dot{\text{D}}$M.[18] This assumption is not important for the results. The techniques developed here can be modified to accommodate a broader set of tie-breaking rules (any representation satisfying Definition 5), but sender-preferred tie-breaking allows a slightly cleaner presentation.

The main result of this section is:

Theorem 4. *The following are equivalent:*

(i) $\nu = \dot{\nu}$, $\mu = \dot{\mu}$, $v \approx \dot{v}$ *and* $u \approx \dot{u}$

(ii) For all menus A, $\succsim^A = \dot{\succsim}^A$

Theorem 4 states that informational preferences $(\succsim^A)_{A\in\mathcal{A}}$, alone, are sufficient to pin down the priors ν, μ and utility indices v, u (up to positive affine transformation). For identification purposes, second-period choice data $(c^s)_{s\in S}$ are not required. This formally expresses the idea that, for Bayesian decision makers, preferences for information are powerful and revealing primitives of analysis.

It is easiest to proceed by breaking the result into a series of smaller steps. I begin by establishing conditions under which DM and $\dot{\text{D}}$M share common first- or second-period priors. With those characterizations in place, I provide a sketch of the proof of Theorem 4 in section 5.2. Given that Theorem 4 is a rather surprising result, section 5.3 provides further discussion of this theorem and related issues.

Some additional terminology is needed to proceed. A *bet* is a menu of the form $A = \{pEq, pFq\}$ where $E, F \subsetneq \Omega$ are nonempty and $E \neq F$. Whenever the need to be explicit about E and F arises, I will refer to such menus as *EF-bets*. Similarly, a bet may be referred to as a *(p, q)-bet*. Bets are useful devices because they are common in real life and because

[18] Recall that under sender-preferred tie-breaking, each $\succsim^A$ is represented by the function $\overline{V}^A(\sigma) := \max_{f\in F^A(\sigma)} \sum_\omega v(f_\omega)\nu_\omega$.

cardinal properties of utility indices do not influence decision making in bets.[19]

For each ω, let $e^\omega \in S$ denote the signal s such that $s_\omega = 1$ and $s_{\omega'} = 0$ for all $\omega' \neq \omega$. Then $\sigma^* := [e^\omega : \omega \in \Omega]$ (the identity matrix) denotes *perfect information*; that is, σ^* reveals the true state ω. Let $e = (1, \ldots, 1) \in S$ denote the signal assigning likelihood 1 to each state of the world. Notice that e itself (interpreted as a column vector) qualifies as an experiment. In particular, e denotes an experiment generating *no information*.

A *neighborhood* of a signal s is a set $N^\varepsilon(s) \subseteq S$ consisting of all signals t such that $\|s - t\| < \varepsilon$ (that is, all signals within distance $\varepsilon > 0$ of s in the standard Euclidean topology).[20]

5.1 Eliciting Priors

When comparing priors, the analysis revolves around the concept of *equivalent signals*. If $s, t \in S$ are distinct signals and σ is an experiment such that $s, t \in \sigma$, let σ^{s+t} denote the experiment formed by deleting column t and replacing column s with $s + t$.[21]

Definition 6. Let $E, F \subsetneq \Omega$ be nonempty events such that $E \neq F$ and let $s, t \in S$.

(i) s and t are *weakly EF-equivalent* if $\sigma \sim^A \sigma^{s+t}$ whenever A is an EF-bet and $s, t \in \sigma$.

(ii) s and t are *EF-equivalent* if there exist neighborhoods $N^\varepsilon(s), N^\varepsilon(t)$ of s and t such that s' and t' are weakly EF-equivalent whenever $s' \in N^\varepsilon(s)$ and $t' \in N^\varepsilon(t)$.

Similar definitions hold for $\dot{E}F$-equivalence by replacing $\sim^A$ with $\dot{\sim}^A$. Signals s and t are *equivalent* if they are both EF- and $\dot{E}F$-equivalent.

The idea of Definition 6 is that EF-equivalent signals yield the same posterior ranking of E and F for DM2: if s and t are EF-equivalent, then $\mu^s(E) > \mu^s(F)$ if and only if $\mu^t(E) > \mu^t(F)$. Since DM2's choices from EF-menus only depend on the posterior rankings, EF-equivalent signals yield the same choices from a given EF-bet. Hence, DM1 will be indifferent between σ and σ^{s+t}. Note that although the definition requires $\sigma \sim^A \sigma^{s+t}$ for all EF-bets A and all σ with $s, t \in \sigma$, in practice a single such indifference likely indicates the desired equivalence; only relatively rare instances, such as when DM2 is indifferent between the lotteries p, q involved in the EF-bet, can yield $\sigma \sim^A \sigma^{s+t}$ for non-EF-equivalent signals.

[19]If, for example, $u(p) > u(q)$, then DM2 prefers pEq over pFq if and only if he assigns greater probability to event E than to event F; the magnitude $u(p) - u(q)$ has no effect.

[20]Note that this differs from the topology on S employed by Axiom B3. For the purposes of this section, the Euclidean topology is easier to work with.

[21]In the event that multiple columns of σ coincide with s or t, take σ^{s+t} to be an experiment formed by deleting one column corresponding to t, and adding t to one column corresponding to s.

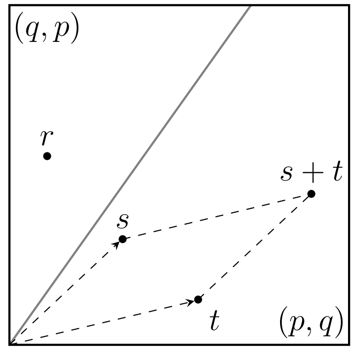


Figure 4: Equivalent signals. Here, s and t are equivalent because they both belong to the (p,q) region of $A = \{(p,q),(q,p)\}$. Thus, so does $s+t$. It follows that $\sigma \sim^A \sigma^{s+t}$, where $\sigma = [r,s,t]$ and $\sigma^{s+t} = [r, s+t]$. Note that r is not equivalent to s or t. The slope of the line separating the (p,q) and (q,p) regions has slope $\frac{\mu_1}{\mu_2}$. Thus, knowing which signals are equivalent reveals μ.

Proposition 1. *The following are equivalent:*

(i) $\mu = \dot{\mu}$

(ii) For all nonempty $E, F \subsetneq \Omega$ *with* $E \neq F$*,* EF*-equivalence implies* $\dot{EF}$*-equivalence.*

(iii) There exists $E, F \subseteq \Omega$ *such that* $E \not\subseteq F$*,* $F \not\subseteq E$*, and* EF*-equivalence implies* $\dot{EF}$*-equivalence.*

Proposition 1 says the receiver's prior can be identified from the sender's preferences for information independently of whether they share a common prior or utility index. Specifically, the set of EF-equivalent signals reveals μ. Part (iii) implies that μ can be identified by determining the set of EF-equivalent signals for a single choice of E and F; there is no need to examine all combinations of E and F.

Next, consider the problem of comparing first-period priors ν and $\dot{\nu}$. The next proposition shows that these priors are characterized by DM1's preferences over a particular class of experiments composed of equivalent signals. Given an EF-bet A, binary experiments $\sigma = [s,t]$, $\sigma' = [s', t']$ are *equivalent* if s is EF-equivalent to s' and t is EF-equivalent to t'.

Proposition 2. *The following are equivalent:*

(i) $\nu = \dot{\nu}$

(ii) For all bets A *and equivalent experiments* σ, σ' *that neither* $\succsim^A$ *nor* $\dot{\succsim}^A$ *rank indifferent to* e*,* $\sigma \sim^A \sigma' \Rightarrow \sigma \mathrel{\dot{\sim}}^A \sigma'$*.*

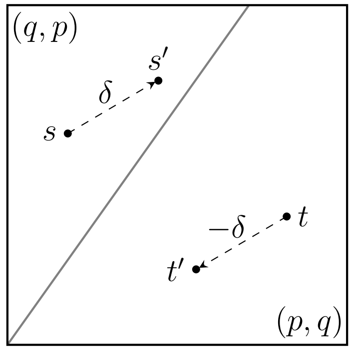


Figure 5: Identifying ν from menu $A = \{(p, q), (q, p)\}$. If $[s, t] \sim^A [s + \delta, t - \delta]$, then $\frac{\delta_1}{\delta_2} = \frac{\nu_2}{\nu_1}$.

Proposition 2 says that the sender's prior is revealed by the indifferences in his preferences for information when facing bets. The idea is that when $\sigma = [s, t]$ and $\sigma' = [s', t']$ are equivalent experiments, there is a $\delta \in \mathbb{R}^\Omega$ such that $\sigma' = [s+\delta, t-\delta]$. Moreover, $c^s(A) = c^{s'}(A)$ and $c^t(A) = c^{t'}(A)$. The fact that neither experiment is ranked indifferent to e implies that $c^s(A) \neq c^t(A)$, as illustrated in Figure 5. The representation, together with $\sigma \sim^A \sigma'$, implies $\sum_{\omega \in E} \nu_\omega \delta_\omega - \sum_{\omega \in E^c} \nu_\omega \delta_\omega = 0$. The set of vectors δ satisfying this expression is revealed by all such indifferences $\sigma \sim^A \sigma'$ and has dimension $|\Omega| - 1$, thus pinning down ν. For example, with $|\Omega| = 2$, ν is pinned down by a single such δ: $\nu_1 \delta_1 = \nu_2 \delta_2$, together with $\nu_1 + \nu_2 = 1$, reveals ν.

5.2 Proof of Theorem 4

In this section, I sketch the main arguments required to prove Theorem 4; for a complete proof, please see the appendix. Clearly, $(\succsim^A)_{A \in \mathcal{A}} = (\dot{\succsim}^A)_{A \in \mathcal{A}}$ if $\nu = \dot{\nu}$, $\mu = \dot{\mu}$, $v \approx \dot{v}$, and $u \approx \dot{u}$. For the converse, suppose $(\succsim^A)_{A \in \mathcal{A}} = (\dot{\succsim}^A)_{A \in \mathcal{A}}$. Then $\nu = \dot{\nu}$ and $\mu = \dot{\mu}$ by Propositions 1 and 2. Thus, the only task is to show that the utility indices are pinned down by the sender's preferences for information.

The first step is to show that the indifference curves for v and u can be identified by examining the sender's preferences in bets. Observe that if A is a (p, q)-bet, then $\succsim^A$ is degenerate ($\sigma \sim^A \sigma'$ for all σ, σ') if and only if $v(p) = v(q)$. Thus, the indifference curves of v in ΔX can be identified from $(\succsim^A)_{A \in \mathcal{A}}$, and they must coincide with those of $\dot{v}$ since $(\succsim^A)_{A \in \mathcal{A}} = (\dot{\succsim}^A)_{A \in \mathcal{A}}$.

Now fix an interior lottery p. For any lottery $q \neq p$, DM1 and DM2 agree on the ranking of p and q if and only if $\sigma^* \succ^A e$, where A is a (p, q)-bet.[22] Thus, the agreement and disagreement regions are revealed by $(\succsim^A)_{A \in \mathcal{A}}$. By linearity, these regions reveal the

[22]Specifically, DM1 and DM2 agree on the ranking of p and q if $v(p) \neq v(q)$ and u (at least weakly) agrees with the ranking of p and q under v.

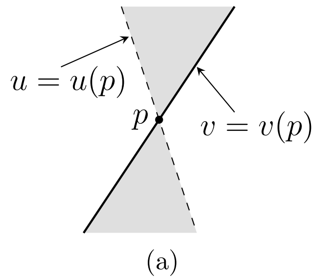


Figure 6: Identifying two candidates for v and u. The (linear) indifference curve for v through p can be identified from $(\succsim^A)_{A\in\mathcal{A}}$, as can the agreement region for u and v (panel (a)); thus, indifference curves for u can be identified. Panels (b) and (c) indicate the two possibilities for the direction of increasing utilities that are consistent with the agreement region (shaded), implying that either (v, u) or $(-v, -u)$ represent preferences. In each case, lottery p is strictly preferred over q.

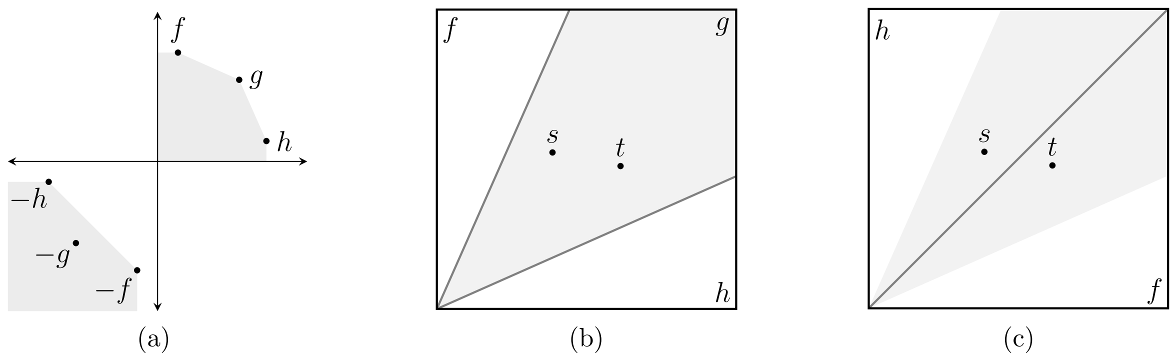


Figure 7: Identifying v and u. If f, g, and h are each chosen by DM2 under (v, u), then only f and h are chosen by DM2 under $(-v, -u)$ (panel (a)). Thus, (v, u) and $(-v, -u)$ yield different divisions of the signal space (panels (b) and (c)). In (b), experiment $\sigma = [s, t]$ is ranked indifferent to e by DM1 because both signals result in choice g. In (c), s and t yield different choices, and therefore (for most choices of f and h) σ is not ranked indifferent to e. Thus, only one of (v, u) or $(-v, -u)$ can be consistent with DM1's preferences $(\succsim^A)_{A\in\mathcal{A}}$.

indifference curves for u (and, once again, these must agree with the indifference curves of $\dot{u}$).

The agreement and disagreement regions narrow the possibilities for v and u down to two cases; specifically, there are indices v and u such that either (v, u) or $(-v, -u)$ are the correct functions (up to positive affine transformation); see Figure 6.

The final step is to show that only one of (v, u) or $(-v, -u)$ can rationalize $(\succsim^A)_{A\in\mathcal{A}}$. The key is to consider a menu $A = \{f, g, h\}$ where, under index u, each act is the unique optimum under some signal. Then, under index $-u$, one act is never selected at any signal. This yields a different division of S, so that (for example) there are experiments σ, σ' that the sender ranks indifferent under $(-v, -u)$ but not under (v, u); see Figure 7.

5.3 Additional Remarks on Theorem 4

Theorem 4 is a strong identification result, and indicates that it may be possible to reformulate (axiomatize) the model using only the preferences $(\succsim^A)_{A\in\mathcal{A}}$. However, Theorem 4 holds under the *assumption* that DM1 and DM2 conform to Value of Information and Bayesian representations, respectively, and it is not clear that such an axiomatization is feasible. Moreover, the characterization of DM2 (Theorem 2) is of independent interest, as it provides a novel characterization of Bayesianism under a rich domain of signals.

Regarding feasibility, there are two issues. First, even with signal-contingent choices, proving the existence of a Value of Information Representation requires considerable effort. Second, and more importantly, it is not clear that suitable axioms on $(\succsim^A)_{A\in\mathcal{A}}$ can be found that fully characterize the representation. For example, it is not obvious how axioms on $(\succsim^A)_{A\in\mathcal{A}}$ can substitute for the Bayesian Independence axiom of DM2, or how the State Independence and Consistency properties of DM1 can be expressed without explicitly referencing second-period choices c^s. It seems unlikely that a characterization can be found that does not begin by deriving candidates for some (possibly intermediate) parameters and then asserting axioms on those derived objects—a less than ideal approach, at least for readers who take a descriptive view of decision theory.

One possibility is to consider a restricted model where the individual is dynamically consistent (that is, $\nu = \mu$ and $v \approx u$). Under these assumptions, preferences $(\succsim^A)_{A\in\mathcal{A}}$ satisfy stronger forms of continuity and independence, as well as the Blackwell information ordering. This seems like a more promising approach, and may be an interesting avenue for future research.

6 Comparing Individuals

In this section, I show how the data $(\succsim^A)_{A\in\mathcal{A}}$ and $(c^s)_{s\in S}$ can be used to make comparisons between the priors and preferences of the sender and receiver. The analysis is divided into two parts. First, in section 6.1, I examine how the sender's preferences for information may be used to test whether the sender and receiver share a common prior, a common utility index, or both. Then, in section 6.2, I develop comparative notions of dynamic (in)consistency. In particular, I explore what it means for the sender and receiver to have priors or utilities that disagree more (or less), and characterize these relationships in terms of the sender's preferences for information.

As in section 5, the emphasis is on the underlying parameters. Thus, the results are relevant for an outsider observer who is willing to assume the axioms are satisfied and wishes to perform comparative statics on the parameters. Throughout, I assume sender-preferred tie-breaking. Once again, this simplifies the presentation but is not essential for the results.

Several results involve the Blackwell information ordering, denoted $\sqsupseteq$. Thus, $\sqsupseteq$ is a partial order on $\mathcal{E}$, and $\sigma \sqsupseteq \sigma'$ means σ is more informative than σ'.[23] As is well-known, $\sigma \sqsupseteq \sigma'$ if and only if σ' is a *garbling* of σ; that is, $\sigma' = \sigma M$, where M is a stochastic matrix (each row of M is a probability distribution). Clearly, $\sigma^* \sqsupseteq \sigma \sqsupseteq e$ for all σ (recall that σ^* denotes perfect information, while e denotes no information).

6.1 Comparing DM1 and DM2

Suppose $(\succsim^A)_{A\in\mathcal{A}}$ and $(c^s)_{s\in S}$ satisfy axioms A1–A6 and B1–B5, so that DM1 has a Value of Information representation (ν, v) and DM2 has a Bayesian representation (μ, u). The goal is to formulate tests indicating whether the two decision makers have a common prior or a common utility index without having to explicitly identify these parameters. The first result provides a characterization of common utility indices.

Proposition 3. *The following are equivalent:*

(i) $v \approx u$

(ii) For all menus A *and all experiments* σ, $\sigma^* \succsim^A \sigma$

(iii) For all bets A *and all experiments* σ, $\sigma^* \succsim^A \sigma$

Proposition 3 offers a simple way to test whether preferences over ΔX are a source of disagreement for the sender and receiver. Specifically, the sender prefers full disclosure in all menus if and only if $v \approx u$. This characterization holds independently of whether the sender and receiver share a common prior. Moreover, the characterization is expressed solely in terms of the informational preferences of the sender—signal-contingent choices of the receiver are not required to test whether $v \approx u$.

To see why Proposition 3 holds, observe that when σ^* reveals the true state ω, DM2 chooses an act $f^* \in A$ such that $u(f^*_\omega) \geq u(f_\omega)$ for all $f \in A$. If $v \approx u$, then DM1 and DM2 agree on the optimal act. This holds for all states, and therefore $\sigma^* \succsim^A \sigma$ for all $\sigma \in \mathcal{E}$. Conversely, if DM1 and DM2 disagree about the ranking of some pair of lotteries p and q, then there is a menu A (in fact, a (p, q)-bet) and a state ω where DM1 disagrees with DM2's choice from A, making full disclosure suboptimal for DM1. For a complete proof, please see the appendix.

The characterization of common priors revolves around the following definition:

[23] By now there are many different presentations of Blackwell's characterization. See de Oliveira (2018), Bielinska-Kwapisz (2003), Crémer (1982), or Leshno and Spector (1992) for accessible treatments.

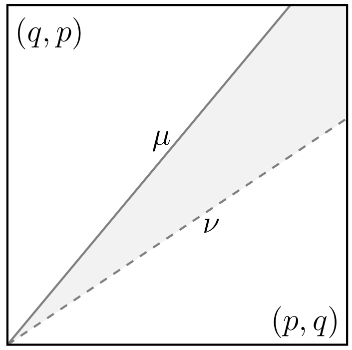


Figure 8: Illustration of Proposition 4. The μ line (slope $\frac{\mu_1}{\mu_2}$) is DM2's cutoff, and the ν line (slope $\frac{\nu_1}{\nu_2}$) is DM1's (hypothetical) cutoff. If DM1 and DM2 agree on the ranking of p and q, then DM1 disagrees with DM2's choices at signals inside the shaded region, and agrees outside of that region (if they disagree on the ranking of p and q, then the agreement and disagreement regions are swapped). When such a wedge exists, Blackwell monotonicity fails if both decision makers are not indifferent between p and q.

Definition 7 (Blackwell Monotonicity)**.** A binary relation $\succsim$ on $\mathcal{E}$ is *Blackwell monotone* if either

(i) For all $\sigma, \sigma' \in \mathcal{E}$, $\sigma \sqsupseteq \sigma'$ implies $\sigma \succsim \sigma'$, or

(ii) For all $\sigma, \sigma' \in \mathcal{E}$, $\sigma \sqsupseteq \sigma'$ implies $\sigma' \succsim \sigma$.

In other words, a preference $\succsim$ is Blackwell monotone if it agrees with the Blackwell ordering (whenever applicable) on all of $\mathcal{E}$, or reverses it (whenever applicable) on all of $\mathcal{E}$.

Proposition 4. *The following are equivalent:*

(i) $\nu = \mu$

(ii) For all bets A, $\succsim^A$ is Blackwell monotone

Proposition 4 establishes Blackwell monotonicity of $\succsim^A$ for all bets A as a necessary and sufficient condition for common priors. This characterization holds independently of whether $v \approx u$ and, like Proposition 3, utilizes only the informational preferences of the sender.[24]

To understand why Proposition 4 holds, suppose there are two states and consider a bet $A = \{(p,q),(q,p)\}$ where $u(p) > u(q)$. Then DM2 chooses (p,q) if and only if $s_1\mu_1 \geq s_2\mu_2$. If DM1 could (hypothetically) choose from A after observing s, and if his preferences satisfied $v(p) > v(q)$, he would choose (p,q) if and only if $s_1\nu_1 \geq s_2\nu_2$ (if instead $v(q) > v(p)$, he

[24]Observe that statement (ii) of Proposition 4 is testable and could be asserted as an additional axiom on $(\succsim^A)_{A\in\mathcal{A}}$. The associated representation theorem would characterize Value of Information representations with common priors.

would choose (p, q) if and only if $s_2\nu_2 \geq s_1\nu_1$). Thus, if $\nu = \mu$ and $v(p) > v(q)$, DM2's choices agree with DM1's hypothetical choices at all signals, but reverses them at all signals if $v(q) > v(p)$. This makes $\succsim^A$ Blackwell monotone (see the appendix for a complete proof).

If, on the other hand, $\nu \neq \mu$, then there is a wedge between the cutoffs for DM1 and DM2 (Figure 8). If DM1 and DM2 agree on the ranking of p and q (that is, either they both prefer p over q or they both prefer q over p), then DM1's (hypothetical) choices agree with those of DM2 outside of the wedge and disagree with those inside the wedge. If they disagree on the ranking of p and q, then the wedge is the agreement region instead. In either case, violations of Blackwell monotonicity emerge by constructing experiments $\sigma \sqsupseteq \sigma'$ utilizing signals near DM2's cutoff. In particular, garbling σ in such a way that a signal crosses over DM2's cutoff can yield the desired violation. The construction is somewhat involved and therefore relegated to the appendix.

Corollary 1. *If $\mu = \nu$ and A is a bet, then either*

(i) $\sigma^* \succsim^A \sigma \succsim^A e$ for all σ, or

(ii) $e \succsim^A \sigma \succsim^A \sigma^*$ for all σ

Corollary 1 follows immediately from Proposition 4 and the fact that $\sigma^* \sqsupseteq \sigma \sqsupseteq e$ for all σ. If A is a (p, q)-bet where the sender and receiver agree on the ranking of p and q, case (i) applies; otherwise, case (ii) applies. This result does not fully characterize common priors, but provides a simple way to test (refute) the hypothesis of common priors: σ^* and e must be at opposite extremes of $\succsim^A$ in order for $\nu = \mu$ to hold.

The decision maker is *dynamically consistent* if $\nu = \mu$ and $v \approx u$. The next proposition is a direct consequence of Propositions 3 and 4.

Proposition 5. *The decision maker is dynamically consistent if and only if either of the following (equivalent) conditions hold:*

(i) For all menus A, $\sigma \sqsupseteq \sigma'$ implies $\sigma \succsim^A \sigma'$

(ii) For all bets A, $\sigma \sqsupseteq \sigma'$ implies $\sigma \succsim^A \sigma'$

This proposition says that the decision maker is dynamically consistent (and therefore behaves like a standard Bayesian) if and only if his informational preferences satisfy the Blackwell information ordering in all decision problems. Part (ii) says that, in fact, adherence to the Blackwell ordering in bets characterizes standard Bayesian behavior.

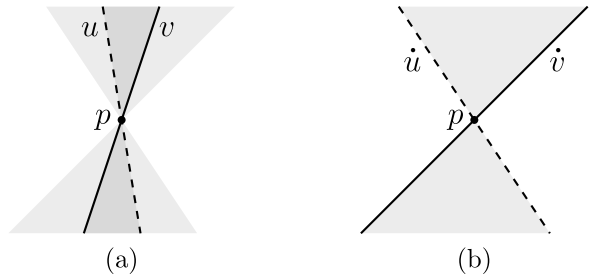


Figure 9: More consistent utilities. In (a), v and u are nearly opposite preferences, as indicated by the narrow agreement region. In (b), the agreement region expands.

6.2 Measures of Consistency

The goal of this section is to define what it means for the attributes (priors and utilities) of the sender to agree more with those of the receiver, and to characterize these relationships in terms of the sender's value of information. Once again, analysis is carried out by considering two pairs of decision makers, DM and $\dot{\text{DM}}$, and I maintain the assumption of sender-preferred tie-breaking for both pairs. Proposition 6 characterizes $\dot{\text{DM}}$ as having first- and second- period utility indices that are more aligned than those of DM, while Proposition 7 characterizes $\dot{\text{DM}}$ as having first- and second-period priors that are more aligned.

Definition 8. DM1 and DM2 *agree on the ranking* of lotteries $p, q \in \Delta X$ if either $[v(p) > v(q)$ and $u(p) \geq u(q)]$ or $[v(q) > v(p)$ and $u(q) \geq u(p)]$. A similar definition holds for $\dot{\text{DM}}$ with $\dot{v}$ and $\dot{u}$ in place of v and u.

Definition 8 says that DM1 and DM2 agree on the ranking of p and q if u does not contradict a strict ranking of p and q under v: if the sender strictly prefers p over q, then the receiver (at least weakly) prefers p over q as well. This leads naturally to the following:

Definition 9. The preferences of $\dot{\text{DM}}$ are *more consistent* than those of DM if, for all $p, q \in \Delta X$ such that DM1 and DM2 agree on the ranking of p and q, $\dot{\text{DM}}1$ and $\dot{\text{DM}}2$ agree on the ranking as well.

Note that Definition 9 does not require DM and $\dot{\text{DM}}$ to rank p and q the same way. For example, DM1 and DM2 may prefer p over q while $\dot{\text{DM}}1$ and $\dot{\text{DM}}2$ prefer q over p. All that matters is that within each pair of decision makers, there is no disagreement regarding the ranking of p and q.

It is not difficult to see that if $\dot{v}$ and $\dot{u}$ are "between" the indices v and u in that $\dot{v} = \alpha v + (1 - \alpha)u$ and $\dot{u} = \beta v + (1 - \beta)u$ for some $\alpha, \beta \in [0, 1]$, then $\dot{v}$ and $\dot{u}$ are more consistent than v and u. Hence, the definition is not vacuous.

Proposition 6. *The following are equivalent:*

(i) The preferences of $\dot{\text{D}}$M are more consistent than those of DM

(ii) For all bets A and experiments σ, $\sigma^* \succ^A \sigma$ implies $\sigma^* \dot{\succ}^A \sigma$

Proposition 6 says that the preferences of $\dot{\text{D}}$M are more aligned than those of DM if, when facing bets, $\dot{\text{D}}$M1 finds perfect information more attractive than DM1 does. The logic of this result is similar to that of Proposition 3. If A is a (p, q)-bet and $\sigma^* \succ^A \sigma$, then in fact $\sigma^* \succsim^A \sigma'$ for all σ'. Thus, DM1 and DM2 agree on the ranking of p and q (in particular, v yields a strict preference that u does not reverse). For $\dot{\text{D}}$M1 and $\dot{\text{D}}$M2 to agree on the ranking of p and q, we therefore require $\sigma^* \dot{\succ}^A \sigma$ (and, hence $\sigma^* \dot{\succsim}^A \sigma'$ for all σ'). Note that $\dot{\text{D}}$M1 and $\dot{\text{D}}$M2 agree on all rankings (that is, $\dot{v} = \dot{u}$) only when perfect information is optimal for $\dot{\text{D}}$M1 in all bets. Thus, Proposition 6 is a natural extension of Proposition 3.

Definition 10. Let $E, F \subseteq \Omega$ and $s \in S$. DM1 and DM2 *agree on the ranking of E and F at s* if either $[\nu^s(E) > \nu^s(F)$ and $\mu^s(E) > \mu^s(F)]$ or $[\nu^s(F) > \nu^s(E)$ and $\mu^s(F) > \mu^s(E)]$. A similar definition holds for $\dot{\text{D}}$M with $\dot{\nu}$ and $\dot{\mu}$ in place of ν and μ.

This definition says that the sender and receiver agree on E and F at s if and only if the Bayesian posteriors ν^s and μ^s rank E and F the same way: either both assign greater probability to E, or both assign greater probability to F.

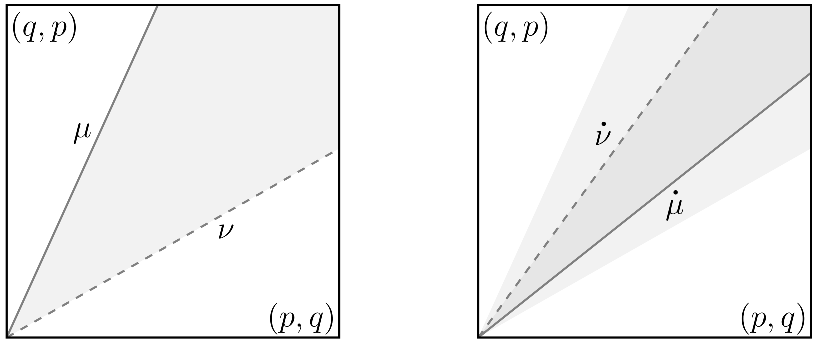
Definition 11. The priors of $\dot{\text{D}}$M are *more consistent* than those of DM if, for all $E, F \subseteq \Omega$ and all $s \in S$, $\dot{\text{D}}$M1 and $\dot{\text{D}}$M2 agree on the ranking of E and F at s whenever DM1 and DM2 agree on the ranking of E and F at s.

In other words, the priors of $\dot{\text{D}}$M are more consistent than those of DM if, for any pair of events, there is a larger set of signals that make $\dot{\text{D}}$M1 and $\dot{\text{D}}$M2 agree on the ranking of those events. In this sense, having more consistent priors means that it is "easier" to get the sender and receiver to agree on the ranking of any pair of events.

Definition 11 is not vacuous: if, for example, $\alpha, \beta \in [0, 1]$, $\dot{\nu} = \alpha\nu + (1 - \alpha)\mu$, and $\dot{\mu} = \beta\nu + (1 - \beta)\mu$, then the priors of $\dot{\text{D}}$M are more consistent than those of DM.[25] Thus, when $\dot{\mu}$ is "closer" to $\dot{\nu}$ than μ is to ν, the priors of $\dot{\text{D}}$M are more consistent.

To relate informational preferences to the notion of more-consistent priors, it is tempting to extend the logic of Proposition 4. For example, if A is a (p, q)-bet and DM1 and DM2 agree on the ranking of p and q, then $\nu = \mu$ if and only if $\succsim^A$ satisfies the Blackwell ordering. Thus,

[25] In fact, with $|\Omega| = 2$, the priors of $\dot{\text{D}}$M are more consistent than those of DM if and only if such α and β exist. With three or more states, $\dot{\nu}$ and $\dot{\mu}$ need not be mixtures of ν and μ in order for the priors of $\dot{\text{D}}$M to be more consistent than those of DM.



(a) (b)

Figure 10: More consistent priors. In (a), there is a relatively large gap between ν and μ—the slopes of the two lines are ν_1/ν_2 and μ_1/μ_2, respectively. In (b), the gap narrows.

if $\nu \neq \mu$, one might expect fewer violations of the Blackwell ordering as the gap between ν and μ closes. Indeed, as the wedge between ν and μ (Figure 10) narrows, the "agreement" region expands and there are new pairs of experiments where $\sigma \sqsupseteq \sigma'$ and $\sigma \mathrel{\dot{\succsim}}^A \sigma'$, making $\dot{\succsim}^A$ more consistent with the Blackwell ordering.

It turns out, however, that if the priors of $\dot{\text{DM}}$ are more consistent than those of DM, then $\dot{\succsim}^A$ is only more consistent with the Blackwell ordering than $\succsim^A$ on a restricted domain of experiments. In particular, a ranking $\sigma \succsim^A \sigma'$ (where $\sigma \sqsupseteq \sigma'$) may be reversed by $\dot{\succsim}^A$ if σ is not "extreme" in the sense defined below.

Some additional definitions are needed to formalize these concepts. An *ε-neighborhood* of an experiment σ is a set

$$N^\varepsilon(\sigma) := \left\{ \sigma' \in \mathcal{E} : \forall s' \in \sigma',\ s' \in \bigcup_{s \in \sigma} N^\varepsilon(s) \right\}$$

where $\varepsilon > 0$ and $N^\varepsilon(s) := \{t \in S : \|s - t\| < \varepsilon\}$. Thus, for every $\sigma' \in N^\varepsilon(\sigma)$ and $s' \in \sigma'$, there is a signal $t \in \sigma$ such that s' is within distance ε of t.

A preference relation $\succsim$ over $\mathcal{E}$ is *degenerate* if $\sigma \sim \sigma'$ for all $\sigma, \sigma' \in \mathcal{E}$; otherwise, it is *non-degenerate*. A menu A is a *non-degenerate bet* if it is a bet and both $\succsim^A$ and $\dot{\succsim}^A$ are non-degenerate.

Definition 12. Let $E, F \subsetneq \Omega$ be nonempty events such that $E \neq F$. An experiment σ is *EF-extreme* if there exists a neighborhood $N^\varepsilon(\sigma)$ such that, for all non-degenerate EF-bets A and all $\sigma' \in N^\varepsilon(\sigma)$ where $\sigma \in \mathcal{E}^*(A)$ and $\sigma \sqsupseteq \sigma'$,

(i) $\sigma \not\sim^A e$

(ii) $\sigma^* \succ^A e \Rightarrow \sigma \succsim^A \sigma'$

(iii) $e \succ^A \sigma^* \Rightarrow \sigma' \succsim^A \sigma$

The definition of $\dot{E}F$-extreme experiments is similar, with $\dot{\succsim}^A$ in place of $\succsim^A$.

Definition 12 says that an experiment σ is EF-extreme if, in every non-degenerate EF-bet $A = \{pEq, pFq\}$, $\succsim^A$ is Blackwell monotone near σ and σ is not ranked indifferent to e. Blackwell monotonicity is imposed by conditions (ii) and (iii), which distinguish whether DM1 and DM2 agree or disagree on the ranking of p and q. Condition (ii) says that if they agree on the ranking, then $\succsim^A$ satisfies the Blackwell ordering on $N^\varepsilon(\sigma)$, while (iii) says that if they disagree on the ranking, then $\succsim^A$ reverses the Blackwell ordering on $N^\varepsilon(\sigma)$.

As shown in the appendix, EF-extremeness amounts to the property that every signal $s \in \sigma$ belongs to the agreement region (if DM1 and DM2 agree on the ranking of p and q), or every $s \in \sigma$ belongs to the disagreement region (if DM1 and DM2 disagree on the ranking of p and q). In both cases, this means the signals of σ are closer to the boundary of S.

Proposition 7. *The following are equivalent:*

(i) The priors of $\dot{D}M$ are more consistent than those of DM

(ii) Every EF-extreme experiment is also $\dot{E}F$-extreme

(iii) If A is an EF-bet, σ is EF-extreme, and $\sigma \sqsupseteq \sigma'$, then $\sigma^* \dot{\succsim}^A e \Rightarrow \sigma \dot{\succsim}^A \sigma'$ and $e \dot{\succsim}^A \sigma^* \Rightarrow \sigma' \dot{\succsim}^A \sigma$.

Proposition 7 provides two characterizations of more consistent priors. To simplify the discussion, I focus here on bets A where both DM and $\dot{D}M$ agree on the ranking of the lotteries. Condition (ii) says that for all E and F, EF-extremeness implies $\dot{E}F$-extremeness. Since extreme experiments consist of signals in the agreement region, this means that $\dot{D}M$ has a larger agreement region, hence more consistent priors.

Condition (iii) says that if σ is EF-extreme and $\sigma \sqsupseteq \sigma'$, then $\sigma \dot{\succsim}^A \sigma'$; that is, $\dot{\succsim}^A$ satisfies the Blackwell ordering on pairs of experiments where the more informative experiment is EF-extreme. This is the sought-after characterization—when priors are more consistent, there are fewer violations of Blackwell monotonicity (provided at least one of the experiments is sufficiently informative).

In practice, an analyst may find it easier to use the identification techniques of section 5 to elicit the parameters (ν, μ) and $(\dot{\nu}, \dot{\mu})$ directly rather than testing the criteria of Proposition 7. Indeed, verifying (or refuting) EF-extremeness requires infinitely many observations if the priors are not known. Nonetheless, Proposition 7 establishes an important qualifier for the intuition that a "more dynamically consistent" individual is more consistent with the Blackwell ordering: at least one experiment must be sufficiently informative, relative to the events under consideration, for the conclusion to hold.

7 Conclusion

In this paper, I have developed a revealed-preference model of information disclosure. Leveraging both the sender's preferences for information and the receiver's signal-contingent choices, the main representation theorem characterizes the testable implications of a large class of communication models with sender commitment power (Bayesian persuasion). An intermediate result characterizes the receiver as a Bayesian information processor, providing a novel foundation for such behavior. The sender and receiver can be interpreted as a single individual, reflecting the behavior of a dynamically inconsistent decision maker who—lacking hard commitment power—influences future choice through selective exposure to new information.

The results highlight the power and usefulness of information structures (Blackwell experiments) as objects of choice. Although information is of purely instrumental value to standard rational decision makers, the sender's preferences for information fully reveal the priors and utility functions of the agents. Testable conditions on the sender's preferences also characterize the differences between the beliefs or utilities of the two decision makers.

An advantage of the informational-preference approach is that it characterizes the interaction in terms of the choices that agents actually make in disclosure models. Moreover, people frequently compare and choose information structures in daily life. The results of this paper demonstrate how observations of such choices can be used to test models and identify parameters, expanding the types of data that can be used in revealed-preference analysis.

It seems plausible that other theories of decision might also be characterized from the perspective of informational choice, and that the techniques developed here can be adapted to that purpose. The standard Bayesian behavior studied here is a natural starting point, but it may (for example) be possible to analyze costly information processing, ambiguity, or other behavioral considerations using preferences for information. I leave the analysis of such models to future work.

A Proof of Theorem 1

Preliminaries

In this section we review some basic definitions and results about affine spaces. Throughout, we work with (nonempty) subsets of $\mathbb{R}^n$.

If $Y \subseteq \mathbb{R}^n$, the *affine hull* of Y is the set

$$\text{aff}(Y) = \left\{ \alpha^0 x^0 + \ldots + \alpha^m x^m : x^0, \ldots, x^m \in Y \text{ and } \sum_{i=1}^{m} \alpha^i = 1 \right\}$$

Elements of $\text{aff}(X)$ are called *affine combinations* of X. Note that the α^i are real numbers (not necessarily belonging to the interval $[0, 1]$). Clearly, $\text{co}(Y) \subseteq \text{aff}(Y)$, where $\text{co}(Y)$ is the convex hull of Y.

A set $Y \subseteq \mathbb{R}^n$ is an *affine space* if $Y = \text{aff}(Y)$. Moreover, every affine space Y is of the form

$$Y = a + Z := \{a + z : z \in Z\}$$

for some $a \in \mathbb{R}^n$ and linear subspace $Z \subseteq \mathbb{R}^n$. Since Z is uniquely determined by Y, we may define the *dimension* of an affine space to be

$$\dim(Y) := \dim(Z),$$

where $Y = a + Z$. We extend this definition to arbitrary convex subsets $C \subseteq \mathbb{R}$ by letting

$$\dim(C) := \dim(\text{aff}(C))$$

That is, the dimension of a convex set is the dimension of its affine hull.

Clearly the set ΔX can be identified with a convex subset of $\mathbb{R}^n$, where $N = |X|$ is the number of outcomes. It is easy to see that $\dim(\Delta X) = |X| - 1$. Similarly, the set of Anscombe-Aumann acts can be identified with the set $\Delta X \times \ldots \times \Delta X = \Delta X^{|\Omega|}$, and has dimension $|\Omega|(N-1)$. We will move freely between the lottery/act and vector representations in several proofs. Finally, we say that a convex subset $C \subseteq (\Delta X)^m$ $(m \geq 1)$ has *full dimension* if $\dim(C) = \dim((\Delta X)^m)$; that is, if $(\Delta X)^m \subseteq \text{aff}(C)$.

A set $\{x^0, \ldots, x^m\} \subseteq \mathbb{R}^n$ is *affinely independent* if $\{x^1 - x^0, \ldots, x^m - x^0\}$ is linearly independent. If $Y \subseteq \mathbb{R}^n$ is an affine space of dimension $m - 1$ and $B = \{x^0, \ldots, x^m\} \subseteq Y$ is affinely independent, then B is an *affine basis* for Y. In that case, every $x \in X$ may be expressed in *affine coordinates*: for each $x \in X$, there are unique scalars $\alpha^0, \ldots, \alpha^m \in \mathbb{R}$

with $\sum \alpha^i = 1$ such that $x = \alpha^0 x^0 + \ldots \alpha^m x^m$. Every affine space has an affine basis.

Let $C \subseteq \mathbb{R}^n$ be convex. A function $T : C \to \mathbb{R}$ is *linear* if $T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y)$ whenever $x, y \in C$ and $\alpha \in [0,1]$. A function $T^* : C \to \mathbb{R}$ is *affine* if

$$T^*(\alpha^0 x^0 + \ldots \alpha^n x^n) = \alpha^0 T^*(x^0) + \ldots + \alpha^n T^*(x^n)$$

whenever $x^i \in C$, $\alpha^0 x^0 + \ldots + \alpha^n x^n \in C$, and $\alpha^0 + \ldots + \alpha^n = 1$. Clearly every affine function is linear; the converse also holds.

If C is convex and $T : C \to \mathbb{R}$ is linear (hence affine), then T has a unique affine extension $T^* : \text{aff}(C) \to \mathbb{R}$. That is, T^* is affine and satisfies $T^*(x) = T(x)$ for all $x \in C$.

Step 1: A linear representation for $\succsim^A$

Lemma 1. *For each menu A and experiment σ, $F^A(\sigma)$ is convex.*

Proof. Suppose $f, g \in F^A(\sigma)$ and $\alpha \in [0,1]$. Then for each $s \in \sigma$ there are acts $f^s, g^s \in \Delta c^s(A)$ such that $f = \left(\sum_{s\in\sigma} s_\omega f^s_\omega\right)_{\omega\in\Omega}$ and $g = \left(\sum_{s\in\sigma} s_\omega g^s_\omega\right)_{\omega\in\Omega}$. Thus

$$\begin{aligned}
\alpha f + (1-\alpha)g &= \left(\alpha \sum_{s\in\sigma} s_\omega f^s_\omega + (1-\alpha)\sum_{s\in\sigma} s_\omega g^s_\omega\right)_{\omega\in\Omega} \\
&= \left(\sum_{s\in\sigma} s_\omega[\alpha f^s_\omega + (1-\alpha)g^s_\omega]\right)_{\omega\in\Omega} \\
&= \left(\sum_{s\in\sigma} s_\omega h^s_\omega\right)_{\omega\in\Omega} \quad \text{where } h^s = \alpha f^s + (1-\alpha)g^s \in \Delta c^s(A),
\end{aligned}$$

so that $\alpha f + (1-\alpha)g \in F^A(\sigma)$. □

Definition 13. For each menu A, let $\mathcal{M}^A := \left\{F^A(\sigma) : \sigma \in \mathcal{E}\right\}$. If $Y, Z \in \mathcal{M}^A$ and $\alpha \in [0,1]$, let $\alpha Y + (1-\alpha)Z := \{\alpha f + (1-\alpha)g : f \in Y, g \in Z\}$.

Lemma 2. *If $\sigma, \sigma' \in \mathcal{E}$ and $\alpha \in [0,1]$, then $F^A(\alpha\sigma \cup (1-\alpha)\sigma') = \alpha F^A(\sigma) + (1-\alpha)F^A(\sigma')$.*

Proof. The statement clearly holds if $\alpha \in \{0,1\}$. So suppose $\alpha \in (0,1)$ and let $\hat{\sigma} = \alpha\sigma \cup (1-\alpha)\sigma'$. Recall that $F^A(\sigma) = \left\{\left(\sum_{s\in\sigma} s_\omega f^s_\omega\right)_{\omega\in\Omega} : f^s \in \Delta c^s(A)\right\}$ and that $F^A(\sigma') = \left\{\left(\sum_{s'\in\sigma'} s'_\omega g^{s'}_\omega\right)_{\omega\in\Omega} : g^{s'} \in \Delta c^{s'}(A)\right\}$. Thus:

$$F^A(\hat{\sigma}) = \left\{\left(\sum_{s\in\sigma} \alpha s_\omega f^t_\omega + \sum_{s'\in\sigma'} (1-\alpha)s'_\omega g^{t'}_\omega\right)_{\omega\in\Omega} : f^t \in \Delta c^{\alpha s}(A), g^{t'} \in \Delta c^{(1-\alpha)s'}(A)\right\}$$

$$= \left\{ \left(\alpha \sum_{s\in\sigma} s_\omega f^t_\omega + (1-\alpha) \sum_{s'\in\sigma'} s'_\omega g^{t'}_\omega \right)_{\omega\in\Omega} : f^t \in \Delta c^{\alpha s}(A), g^{t'} \in \Delta c^{(1-\alpha)s'}(A) \right\}$$

Note that $c^{\lambda t} = c^t$ for all signals t and scalars $\lambda > 0$ because DM2 has a Bayesian representation. Thus

$$\begin{aligned} F^A(\hat\sigma) &= \left\{ \left(\alpha \sum_{s\in\sigma} s_\omega f^s_\omega + (1-\alpha) \sum_{s'\in\sigma'} s'_\omega g^{s'}_\omega \right)_{\omega\in\Omega} : f^s \in \Delta c^{s}(A), g^{s'} \in \Delta c^{s'}(A) \right\} \\ &= \left\{ \alpha f + (1-\alpha) g : f \in F^A(\sigma), g \in F^A(\sigma') \right\}, \end{aligned}$$

as desired. □

Definition 14 (Mixture Space)**.** A *mixture space* (Herstein and Milnor (1953)) is a set $\mathcal{M}$ and an operator $\oplus : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (where $\oplus(\alpha, m, m')$ is written $\alpha m \oplus (1-\alpha)m'$) such that:

(i) $1m \oplus 0m' = m$,

(ii) $\alpha m \oplus (1-\alpha)m' = (1-\alpha)m' \oplus \alpha m$, and

(iii) $\alpha[\beta m \oplus (1-\beta)m'] \oplus (1-\alpha)m' = (\alpha\beta)m \oplus (1-\alpha\beta)m'$.

Lemma 3. *For each menu A, the pair $\left(\mathcal{M}^A, +\right)$ (where $+$ is given by Definition 13) is a mixture space.*

Proof. First, we verify that the family $\mathcal{M}^A$ is closed under the proposed mixture operation. If $Y, Z \in \mathcal{M}^A$, then there exist $\sigma^Y, \sigma^Z \in \mathcal{E}$ such that $F^A(\sigma^Y) = Y$ and $F^A(\sigma^Z) = Z$. Let $\alpha \in [0,1]$. To see that $\alpha Y + (1-\alpha)Z \in \mathcal{M}^A$, apply Lemma 2 to get $F^A(\alpha\sigma^Y \cup (1-\alpha)\sigma^Z) = \alpha F^A(\sigma^Y) + (1-\alpha)F^A(\sigma^Z) = \alpha Y + (1-\alpha)Z$.

The remainder of the argument is standard and well-known, but reproduced here for completeness. Properties (i) and (ii) are simple to verify. For (iii), let $\tilde{Z} = \beta Y + (1-\beta)Z$. To see that $\alpha\tilde{Z} + (1-\alpha)Z = \alpha\beta Y + (1-\alpha\beta)Z$, observe that if $h \in \alpha\tilde{Z} + (1-\alpha)Z$, then there are acts $f \in Y$ and $g, g' \in Z$ such that

$$\begin{aligned} h &= \alpha(\beta f + (1-\beta)g) + (1-\alpha)g' \\ &= \alpha\beta f + \alpha(1-\beta)g + (1-\alpha)g' \\ &= \alpha\beta f + (1-\alpha\beta)\left[\frac{\alpha(1-\beta)}{1-\alpha\beta} g + \frac{1-\alpha}{1-\alpha\beta} g'\right] \\ &\in \alpha\beta Y + (1-\alpha\beta)Z \end{aligned}$$

Conversely, if $h \in \alpha\beta Y + (1-\alpha\beta)Z$, then there are acts $f \in Y$, $g \in Z$ such that

$$\begin{aligned} h &= \alpha\beta f + (1-\alpha\beta)g \\ &= \alpha\beta f + \alpha(1-\beta)g + (1-\alpha)g \\ &= \alpha(\beta f + (1-\beta)g) + (1-\alpha)g \\ &\in \alpha\tilde{Z} + (1-\alpha)Z \end{aligned}$$

Hence, $(\mathcal{M}^A, +)$ is a mixture space. □

Lemma 4. *Every $\succsim^A$ has a unique (up to positive affine transformation) linear representation $V^A : \mathcal{E} \to \mathbb{R}$.*

Proof. The function F^A maps $\succsim^A$ to a complete and transitive relation $\trianglerighteq^A$ on $\mathcal{M}^A$ defined by:

$$Y \trianglerighteq Z \Leftrightarrow \exists \sigma^Y, \sigma^Z \text{ such that } F^A(\sigma^Y) = Y, F^A(\sigma^Z) = Z, \text{ and } \sigma^Y \succsim^A \sigma^Z$$

This is well-defined because Consistency (A6) forces $\sigma \sim^A \sigma'$ whenever $F^A(\sigma) = F^A(\sigma')$. Thus, the induced ranking of Y and Z does not depend on the choice of representatives σ^Y, σ^Z. Clearly every $Y \in \mathcal{M}^A$ has such a representative σ^Y (recall that $\mathcal{M}^A := \{F^A(\sigma) : \sigma \in \mathcal{E}\}$), and completeness and transitivity of $\trianglerighteq^A$ is inherited from $\succsim^A$. Let $\triangleright^A$ denote the strict part of $\trianglerighteq^A$.

By Lemma 2 and the Independence Axiom (A2), $\trianglerighteq^A$ satisfies the standard vNM independence axiom: if $Y \triangleright^A Z$ and $Z' \in \mathcal{M}^A$, then $\sigma^Y \succ^A \sigma^Z$ for all representatives σ^Y, σ^Z of Y and Z. Axiom A2 implies $\alpha\sigma^Y \cup (1-\alpha)\sigma^{Z'} \succ^A \alpha\sigma^Z \cup (1-\alpha)\sigma^{Z'}$ for all $\alpha \in (0,1)$ and all representatives $\sigma^{Z'}$ of Z'. Apply Lemma 2 and the definition of $\trianglerighteq^A$ to get $\alpha Y + (1-\alpha)Z' \triangleright^A \alpha Z + (1-\alpha)Z'$, as desired.

A similar argument employing Lemma 2 and Axiom A3 establishes that $\trianglerighteq^A$ satisfies vNM Continuity: $Y \triangleright^A Z \triangleright^A Z'$ implies there are $\alpha, \beta \in (0,1)$ such that $\alpha Y + (1-\alpha)Z' \triangleright^A Z \triangleright^A \beta Y + (1-\beta)Z'$.

Thus, $\trianglerighteq^A$ is a preference relation satisfying the vNM axioms on the mixture space $(\mathcal{M}^A, +)$. By the Mixture Space Theorem (Herstein and Milnor (1953)), $\trianglerighteq^A$ has a unique (up to positive affine transformation) linear representation $W^A : \mathcal{M}^A \to \mathbb{R}$. This induces a linear representation $V^A : \mathcal{E} \to \mathbb{R}$ for $\succsim^A$ by defining $V^A(\sigma) := W^A(F^A(\sigma))$. Moreover, V^A satisfies

$$\begin{aligned} V^A(\alpha\sigma \cup (1-\alpha)\sigma') &= W^A(F^A(\alpha\sigma \cup (1-\alpha)\sigma')) \\ &= W^A(\alpha F^A(\sigma) + (1-\alpha)F^A(\sigma')) \quad \text{(by Lemma 2)} \\ &= \alpha W^A(F^A(\sigma)) + (1-\alpha)W^A(F^A(\sigma')) \quad \text{(by linearity of } W^A\text{)} \end{aligned}$$

$$= \alpha V^A(\sigma) + (1-\alpha)V^A(\sigma'),$$

so that V^A is a linear representation for $\succsim^A$. □

Step 2: Construction of candidate representation

Recall that $N = |X|$ and u, μ denote the (non-constant) utility index and (full support) prior, respectively, for DM2. This step of the proof constructs a menu A^* where the associated set of induced acts is rich enough to pin down candidates for ν and v.

Lemma 5. *There exists an affinely independent set $P = \{p^1, \ldots, p^N\}$ of interior lotteries such that:*

(i) $u(p^N) > u(p^{N-1}) > \ldots > u(p^1)$, *and*

(ii) $u(p^2) - u(p^1) > u(p^3) - u(p^2) > \ldots > u(p^N) - u(p^{N-1})$.

Proof. It is easy to find interior lotteries satisfying conditions (i) and (ii) (just choose N utility levels in the range of u that satisfy (i) and (ii), and then pick lotteries yielding those utility values). If necessary, perturb these lotteries along indifference curves (hyperplanes) in ΔX to arrive at an affinely independent set. Such perturbations are possible because N lotteries in ΔX fail to be affinely independent if and only if they sit on an $(N-2)$-dimensional hyperplane in ΔX. Since indifference curves are linear and the lotteries are interior, one lottery can be perturbed along its indifference plane to yield an affinely independent set. □

For the remainder of the proof, let $P = \{p^1, \ldots, p^N\}$ satisfy the requirements of Lemma 5. The convex hull $\text{co}(P)$ has dimension $N-1$ (full dimension in ΔX) because P is affinely independent. It will be useful to think of $\text{co}(P)$ as a polytope and each p^i as a vertex of the polytope. Every nonempty $P' \subseteq P$ corresponds to a face of the polytope—in particular, the convex hull $\text{co}(P')$ yields a face of dimension $|P'| - 1$.

Lemma 6. *Let $D \subseteq \Delta X$ be a convex subset of $co(P')$ for some $P' \subseteq P$ such that* $\dim D = \dim co(P')$. *If $\hat{p} \in P \backslash P'$ and $q^1, \ldots, q^n \in co(P' \cup \{\hat{p}\}) \backslash co(P')$, then*

$$\dim \bigcap_{i=1}^{n} co(D \cup \{q^i\}) = \dim D + 1$$

Proof. First, we prove the following claim: if $x \in D$, $q \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$, and $\varepsilon > 0$, then $\dim(\text{co}(D \cup \{q\}) \cap N^\varepsilon(x)) = \dim D + 1$, where $N^\varepsilon(x)$ is the ε-neighborhood of x.

To prove the claim, note that $\dim(\text{co}(D \cup \{q\})) = \dim D + 1$. Therefore, there exists $z^1, \ldots, z^K \in \text{co}(D \cup \{q\}) \backslash \{x\}$ ($K = \dim D$) such that the set $\{x, z^1, \ldots, z^K\}$ is affinely independent. Thus, the set $\{z^1 - x, \ldots, z^K - x\}$ is linearly independent. Clearly, for every $i = 1, \ldots, K$, the line L^i through x and z^i passes through $N^\varepsilon(x)$. For each i, let $x^i \in N^\varepsilon(x) \backslash \{x\}$ be a point on L^i. Then, since $\{z^1 - x, \ldots, z^K - x\}$ is linearly independent, the set $\{x, x^1, \ldots, x^K\}$ is affinely independent. It follows that $\text{co}\{x, x^1, \ldots, x^K\} \subseteq N^\varepsilon(x)$ has dimension $\dim D + 1$, and therefore $N^\varepsilon(x) \cap \text{co}(D \cup \{q\})$ has dimension $\dim D + 1$. This proves the claim.

Now fix a point x in the (relative) interior of D and apply the claim to each $q = q^1, \ldots, q^W \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$. Since x is in the interior of D, there exists $\varepsilon_i > 0$ $(i = 1, \ldots, n)$ such that $N^{\varepsilon_i}(x) \cap \text{co}(D \cup \{\hat{p}\}) = N^{\varepsilon_i}(x) \cap \text{co}(D \cup \{q^i\})$. Let ε denote the smallest such ε_i and choose a point y in the relative interior of $\text{co}(P' \cup \{\hat{p}\}) \cap N^\varepsilon(x)$. By the claim, each set $N^\varepsilon(x) \cap \text{co}(D \cup \{q^i\})$ has dimension $\dim D + 1$, and $y \in \text{co}(P' \cup \{\hat{p}\}) \backslash \text{co}(P')$. Since $y \in \bigcap_{i=1}^n \text{co}(D \cup \{q^i\})$, it follows that $\dim \bigcap_{i=1}^n \text{co}(D \cup \{q^i\})$ has dimension $D + 1$. $\square$

For an ordered pair $E = [\omega, \omega']$ (where $\omega \neq \omega'$), lotteries p, q, and an act h, let $(p, q)Eh$ denote the act f such that $f_\omega = p$, $f_{\omega'} = q$, and $f_{\hat{\omega}} = h_{\hat{\omega}}$ for all $\hat{\omega} \neq \omega, \omega'$. Similar notation applies for signals: if $\alpha, \beta \in [0, 1]$ and $t \in S$, then $(\alpha, \beta)Et$ denotes the profile r where $r_\omega = \alpha$, $r_{\omega'} = \beta$, and $r_{\hat{\omega}} = t_{\hat{\omega}}$ for all $\hat{\omega} \notin E$. To qualify as a signal, at least one entry of r must be nonzero.

Definition 15 (Symmetric Menu)**.** Suppose $P = \{p^1, \ldots, p^N\}$ satisfies the requirements of Lemma 5 and that $u(p) > u(\underline{p})$ for all $p \in P$. For each $E = [\omega, \omega']$, let

$$\begin{aligned} A^E &:= \left\{(p^i, p^{N-i+1})E\underline{p} : i = 1, \ldots, N\right\} \\ &= \left\{(p^1, p^N)E\underline{p}, (p^2, p^{N-1})E\underline{p}, \ldots, (p^N, p^1)E\underline{p}\right\} \end{aligned}$$

and let

$$A^* := \bigcup_E A^E$$

Then A^* is the *symmetric menu on* $(P, \underline{p})$.

Note that a lottery $\underline{p}$ satisfying $u(p) > u(\underline{p})$ for all $p \in P$ exists because each lottery of P is interior. Throughout the remainder of the proof, we take as given a menu A^* satisfying the requirements of Definition 15.

Definition 16. For $E = [\omega, \omega']$ (where $\omega \neq \omega'$) let

$$S^E := \{s \in S : \hat{\omega} \notin E \Rightarrow s_{\hat{\omega}} = 0\}$$

and

$$\mathcal{E}^E := \{\sigma \in \mathcal{E} : \forall s \in \sigma, \text{ either } s \in S^E \text{ or } s = \lambda e^{\hat{\omega}} \text{ for some } \lambda \in (0,1] \text{ and } \hat{\omega} \in \Omega\}$$

Definition 16 says that if $s \in S^E$, then states outside of E are assigned likelihood 0 by s. An experiment $\sigma \in \mathcal{E}^E$ is composed of signals from S^E as well as (scalar multiples of) indicator signals $e^{\hat{\omega}}$ for each $\hat{\omega} \notin E$ (recall that $e^{\hat{\omega}}$ is a signal assigning likelihood 1 to state $\hat{\omega}$ and 0 to all other states). Observe that S^E is convex ($s, t \in S^E$ implies $\alpha s + (1-\alpha)t \in S^E$ for all $\alpha \in [0,1]$) and that $\mathcal{E}^E$ is convex as well ($\sigma, \sigma' \in \mathcal{E}^E$ implies $\alpha\sigma \cup (1-\alpha)\sigma' \in \mathcal{E}^E$ for all $\alpha \in [0,1]$). It is also easy to verify that if $s \in S^E$, then $c^s(A^*) \subseteq A^E$.

Lemma 7. *For each $E = [\omega, \omega']$ and $f \in A^E$, there is an $s \in S^E$ such that $c^s(A^*) = f$.*

Proof. As noted above, we have $c^s(A^*) \subseteq A^E$ whenever $s \in S^E$. Therefore, we only need to show that for each $f \in A^E$, there is a signal $s \in S^E$ such that $f \succsim^s g$ for all $g \in A^E$.

First, we prove that if $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p}$ for some $s \in S^E$, then $(p^{i+1}, p^{N-(i+1)+1})E\underline{p} \succ^s (p^{i+2}, p^{N-(i+2)+1})E\underline{p}$.

If $s \in S^E$ and $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p}$, then

$$s_\omega \mu_\omega u(p^i) + s_{\omega'}\mu_{\omega'} u(p^{N-i+1}) > s_\omega \mu_\omega u(p^{i+1}) + s_{\omega'}\mu_{\omega'} u(p^{N-(i+1)+1})$$

Equivalently,

$$s_{\omega'}\mu_{\omega'}[u(p^{N-i+1}) - u(p^{N-(i+1)+1})] > s_\omega \mu_\omega [u(p^{i+1}) - u(p^i)]$$

Observe that, by our choice of P,

$$u(p^{i+1}) - u(p^i) > u(p^{i+2}) - u(p^{i+1}) \tag{7}$$

and

$$u(p^{N-(i+1)+1}) - u(p^{N-(i+2)+1}) > u(p^{N-i+1}) - u(p^{N-(i+1)+1}) \tag{8}$$

Thus,

$$\begin{aligned} s_{\omega'}\mu_{\omega'}[u(p^{N-(i+1)+1}) - u(p^{N-(i+2)+1})] &> s_{\omega'}\mu_{\omega'}[u(p^{N-i+1}) - u(p^{N-(i+1)+1})] \\ &> s_\omega \mu_\omega [u(p^{i+1}) - u(p^i)] \\ &> s_\omega \mu_\omega [u(p^{i+2}) - u(p^{i+1})] \end{aligned}$$

so that $(p^{i+1}, p^{N-(i+1)+1})E\underline{p} \succ^s (p^{i+2}, p^{N-(i+2)+1})E\underline{p}$.

A similar argument establishes that if $(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i-1}, p^{N-(i-1)+1})E\underline{p}$ for some $s \in S^E$, then $(p^{i-1}, p^{N-(i-1)+1})E\underline{p} \succ^s (p^{i-2}, p^{N-(i-2)+1})E\underline{p}$.

Thus, for $1 < i < N$, we have $c^s(A^*) = (p^i, p^{N-i+1})E\underline{p}$ if and only if

$$(p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i+1}, p^{N-(i+1)+1})E\underline{p} \quad \text{and} \quad (p^i, p^{N-i+1})E\underline{p} \succ^s (p^{i-1}, p^{N-(i-1)+1})E\underline{p}$$

Since $s \in S^E$, it cannot be the case that both $s_\omega = 0$ and $s_{\omega'} = 0$. Suppose $s_{\omega'} > 0$. Using the representation for DM2, the above conditions are equivalent to

$$\frac{\mu_{\omega'}}{\mu_\omega}\frac{u(p^{N-(i-1)+1}) - u(p^{N-i+1})}{u(p^i) - u(p^{i-1})} < \frac{s_\omega}{s_{\omega'}} < \frac{\mu_{\omega'}}{\mu_\omega}\frac{u(p^{N-i+1}) - u(p^{N-(i+1)+1})}{u(p^{i+1}) - u(p^i)}$$

By (7) and (8) (with $i-1$ in place of i), this yields an interval of values for $\frac{s_\omega}{s_{\omega'}}$ such that $c^s(A^*) = (p^i, p^{N-i+1})E\underline{p}$.

For $i = 1$ or $i = N$, observe that $s \in S^E$ satisfies $c^s(A^*) = (p^1, p^N)E\underline{p}$ if and only if $(p^1, p^N)E\underline{p} \succ^s (p^2, p^{N-1})E\underline{p}$ while $c^s(A^*) = (p^N, p^1)E\underline{p}$ if and only if $(p^N, p^1)E\underline{p} \succ^s (p^{N-1}, p^2)E\underline{p}$ (this follows from the first two claims established in this proof). Using the representation for DM2 in a similar manner, it is easy to see that there exist signals $s \in S^E$ such that $c^s(A^*) = (p^1, p^N)E\underline{p}$. □

Lemma 8. *There exists a convex, full-dimensional set $D \subseteq co(P)$ such that, for every $p \in D$ and every state ω, there is an experiment σ such that $F^{A^*}_\omega(\sigma) = p$.*

Proof. We will construct D in several steps. First, enumerate $\Omega = \{1, \ldots, W\}$. We will work with pairs of the form $E = [1, \omega]$ for $\omega = 2, \ldots, W$.

Consider $E = [1, \omega]$. Under perfect information σ^*, we have $F^{A^*}_{\hat\omega}(\sigma^*) = p^N$ for all $\hat\omega$. Notice that $\sigma^* \in \mathcal{E}^E$. There exists $\delta > 0$ such that for $s = (1-\delta, \delta)E0$ and $t = (\delta, 1-\delta)E0$, we have $c^s(A^*) = (p^N, p^1)E\underline{p}$ and $c^t(A^*) = (p^1, p^N)E\underline{p}$. Thus, the experiment $\sigma = [s, t] \cup [e_{\hat\omega} : \hat\omega \notin E]$ yields $F^{A^*}(\sigma) = (\delta p^1 + (1-\delta)p^N, \delta p^1 + (1-\delta)p^N)Ep^N$. Thus, both $F^{A^*}_1(\sigma)$ and $F^{A^*}_\omega(\sigma)$ are $\delta p^1 + (1-\delta)p^N$. Mixing σ^* with σ yields a convex set $D^1_\omega \subseteq \text{co}(\{p^1, p^N\})$ of dimension 1 such that, for all $p \in D^1_\omega$, there exists σ such that $F^{A^*}_{\omega'}(\sigma) = p$ for $\omega' = 1, \omega$. Since every such set lies on the face $\text{co}(\{p^1, p^N\})$ and contains p^N, the set $D^1 := \bigcap_{\omega \geq 2} D^1_\omega$ is nonempty and has dimension 1.

For each $E = [1, \omega]$, pick a signal $s \in S^E$ such that $c^s(A^*) = [p^{N-1}, p^2]E\underline{p}$ (Lemma 7) and an experiment $\sigma \in \mathcal{E}^E$ such that $F^{A^*}_\omega(\sigma) = p$ for some p in the interior of D^1. Thus, σ contains a signal t such that $c^t(A^*)$ is a singleton and $t_{\hat\omega} > 0$ for $\hat\omega = 1, \omega$ (otherwise, p is not in the interior of D^1). Therefore, we may assume $t - s$ is a well-defined signal and that $c^{t-s}(A^*) = c^t(A^*)$ (if necessary, replace s with λs for a sufficiently small $\lambda > 0$). Let σ' denote the experiment formed by taking σ, appending s, and replacing t with $t - s$. Then

$q^\omega := F_\omega^{A^*}(\sigma') \in \text{co}\{p^1, p^2, p^N\} \backslash \text{co}\{p^1, p^N\}$. Taking mixtures of σ' and σ (and letting σ vary in order to generate $F_\omega^{A^*}(\sigma) = p$ for all $p \in \text{int}D^1$) implies that every $q \in \text{co}\{\text{int}D^1, q^\omega\}$ satisfies $F_\omega^{A^*}(\sigma'')$ for some $\sigma'' \in S^E$. Repeating this procedure for every choice of $E = [1, \omega]$ (and also for $E = [\omega, 1]$ for some ω) yields lotteries $q^1, \ldots, q^W \in \text{co}\{p^1, p^2, p^N\} \backslash \text{co}\{p^1, p^N\}$. By Lemma 6, $D^2 := \bigcap_{\omega=1}^W \text{co}(\text{int}D^1 \cup \{q^\omega\})$ has dimension 2. By construction, for every $p \in D^2$ and every ω there is an experiment $\sigma \in \mathcal{E}^E$ such that $F_\omega^{A^*}(\sigma) = p$.

We now proceed by induction. Suppose $D^i \subseteq \text{co}\{p^1, \ldots, p^i, p^N\}$ $(2 \leq i < N)$ is a convex set of dimension i and, for all $p \in D^i$ and all ω, there exists $\sigma \in \mathcal{E}^E$ such that $F_\omega^{A^*}(\sigma) = p$. We construct a convex set $D^{i+1} \subseteq \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\}$ of dimension $i+1$ such that, for every $p \in D^{i+1}$ and ω, there exists $\sigma \in \mathcal{E}^E$ such that $F_\omega^{A^*}(\sigma) = p$. The procedure is similar to the previous step. First, take $E = [1, \omega]$ and an experiment $\sigma \in \mathcal{E}^E$ such that $F_\omega^{A^*}(\sigma) = p$ for some p in the interior of D^i. Then σ contains a signal t such that $c^t(A^*)$ is a singleton and $t_{\hat{\omega}} > 0$ for $\hat{\omega} \in E$. Pick a signal $s \in S^E$ such that $c^s(A^*) = (p^{N-i}, p^{i+1})E\underline{p}$. We may assume that $t - s$ is a well-defined signal such that $c^{t-s}(A^*) = c^t(A^*)$ (if necessary, scale s down by a factor $\lambda > 0$). Let σ' be an experiment formed by deleting t from σ and appending $t - s$ and s. Then $q^\omega := F_\omega^{A^*}(\sigma') \in \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\} \backslash \text{co}\{p^1, \ldots, p^i, p^N\}$. Repeating this for all ω (as well as $E = [1, \omega]$ for some ω) yields lotteries $q^1, \ldots, q^W \in \text{co}\{p^1, \ldots, p^i, p^{i+1}, p^N\} \backslash \text{co}\{p^1, \ldots, p^i, p^N\}$. By Lemma 6, $D^{i+1} := \bigcap_{\omega=1}^W \text{co}(\text{int}D^i \cup \{q^\omega\})$ has dimension $i+1$. By construction, for every $p \in D^{i+1}$ and every ω there is an experiment $\sigma \in \mathcal{E}^E$ such that $F_\omega^{A^*}(\sigma) = p$. □

For the remainder of Step 2 of the proof, let $D \subseteq \text{co}(P)$ be a set satisfying all requirements of Lemma 8. The plan is to pick an interior lottery $p^* \in D$ such that, in each state ω, a full-dimensional set around p^* can be induced in menu A^* while holding the induced lotteries in other states fixed. This will ensure that F^{A^*} has full dimension and that $\succsim^{A^*}$ (restricted to $\mathcal{E}^*(A^*)$) derives from a linear ordering $\succsim$ on F satisfying the standard State Independence axiom.

Definition 17 (Interior Experiment)**.** Fix a menu A. For each $f \in A$, let $S^A(f) := \{s \in S : c^s(A) = f\}$. An experiment σ is *A-interior* if:

(i) $c^s(A)$ is single-valued for all $s \in \sigma$, and

(ii) For each $f \in A$, there is exactly one $s \in \sigma$ such that $c^s(A) = f$.

Similarly, any set σ of signals (not necessarily qualifying as an experiment) is A-interior if it satisfies conditions (i) and (ii) of Definition 17. Such a set is necessarily nonempty and finite.

Let S^* denote the set of all signals s such that $s_\omega > 0$ for all ω. The statement $\sigma \subseteq S^*$ means σ is a matrix where each column is a member of S^*. Note that such matrices do not necessarily qualify as experiments.

Definition 18 (ε-Neighborhood)**.** Suppose $\sigma \subseteq S^*$ is A-interior and let $\varepsilon > 0$. For each $s \in \sigma$, let $Q^{s,\varepsilon} := \prod_\omega (s_\omega - \varepsilon, s_\omega + \varepsilon)$. Let B^ε denote the set of all A-interior matrices $\sigma' \subseteq S^*$ such that:

(i) For each ω, $\sum_{s' \in \sigma'} s'_\omega = \sum_{s \in \sigma} s_\omega$, and

(ii) If $s \in \sigma$, $s' \in \sigma'$, and $c^s(A) = c^{s'}(A)$, then $s' \in Q^{s,\varepsilon}$.

Then B^ε is an *ε-neighborhood* of σ (in A).

Note that Definition 18 does not require σ to be an experiment, and that $B^\varepsilon \subseteq \mathcal{E}$ (in fact, $B^\varepsilon \subseteq \mathcal{E}^*(A)$) if and only if σ is an experiment.

The next two lemmas provide general results about menus and (neighborhoods of) experiments that induce full-dimensional sets of acts. These will be used in Step 3 of the proof as well.

Lemma 9. *Suppose that, for each ω, $L^*_\omega \subseteq \Delta X$ is full-dimensional. Let $f^* \in F$ and define $L^*_\omega[\omega]f^* := \{p[\omega]f^* : p \in L^*_\omega\}$. If $G \subseteq F$ is convex and $L^*_\omega[\omega]f^* \subseteq G$ for all ω, then G has full dimension.*

Proof. We need to show that every Anscombe-Aumann act is in the affine hull of G. To begin, note that for each ω, aff(G) contains aff$(L^*_\omega[\omega]f^*) = \{p[\omega]f^* : p \in \Delta X\}$ since L^*_ω has full dimension in ΔX. Therefore aff(G) contains aff(C), where

$$C = \bigcup_\omega \{p[\omega]f^* : p \in \Delta X\}$$

So, it is enough to find a finite set $B \subseteq C$ such that $F \subseteq \text{aff}(B)$. A natural candidate for B involves the (affinely independent) set $P = \{p^1, \dots, p^N\} \subseteq \Delta X$. In particular, let

$$B = \bigcup_{\omega \in \Omega} \left\{p^i[\omega]f^* : i = 1, \dots, N\right\}$$

Clearly $B \subseteq C$. To see that $F \subseteq \text{aff}(B)$, let $f \in F$ and $\alpha = (\alpha_\omega)_{\omega \in \Omega} \in (0,1)^\Omega$ such that $\sum_\omega \alpha_\omega = 1$. For each ω, we have

$$\frac{\sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega}{1 - \alpha_\omega} = f^*_\omega \in \Delta X$$

Therefore, there is some $\hat{f}_\omega \in \Delta X$ such that

$$\begin{aligned} f_\omega &= \alpha_\omega \hat{f}_\omega + (1-\alpha_\omega)\frac{\sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega}{1-\alpha_\omega} \\ &= \alpha \hat{f}_\omega + \sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega \end{aligned}$$

Since P is affinely independent with $\dim(\text{aff}(P)) = \dim(\Delta X)$, for each ω there are numbers β^i_ω $(i = 1, \ldots, N)$ such that

$$\hat{f}_\omega = \sum_{i=1}^N \beta^i_\omega p^i \text{ and } \sum_{i=1}^N \beta^i_\omega = 1$$

Thus

$$\begin{aligned} f_\omega = \alpha_\omega \hat{f}_\omega + \sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega = \alpha_\omega \sum_{i=1}^N \beta^i_\omega p^i + \sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega \\ = \sum_{i=1}^N \alpha^i_\omega p^i + \sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega, \text{ where } \alpha^i_\omega := \alpha_\omega \beta^i_\omega \end{aligned}$$

Note that $\sum_{i=1}^N \alpha^i_\omega = \alpha_\omega$ for each ω, so that $\sum_\omega \sum_{i=1}^N \alpha^i_\omega = \sum_\omega \alpha_\omega = 1$. Then

$$\begin{aligned} \sum_\omega \sum_{i=1}^N \alpha^i_\omega p^i[\omega] f^* &= \left(\sum_{i=1}^N \alpha^i_\omega p^i + \sum_{\omega' \neq \omega} \sum_{i=1}^N \alpha^i_{\omega'} f^*_\omega \right)_{\omega \in \Omega} \\ &= \left(\alpha_\omega \hat{f}_\omega + \sum_{\omega' \neq \omega} \alpha_{\omega'} f^*_\omega \right)_{\omega \in \Omega} \\ &= f \end{aligned}$$

Thus, $f \in \text{aff}(B)$, as desired. □

Lemma 10. *Suppose $\sigma \in \mathcal{E}$ is A-interior and that, for each ω, there is a nonempty $B \subseteq A$ such that $|B| = N$ and $B_\omega := \{f_\omega : f \in B\}$ is affinely independent. If B^ε is an ε-neighborhood for σ, then:*

(i) For each ω, $F^A(B^\varepsilon) := \{F^A(\sigma') : \sigma' \in B^\varepsilon\}$ has a subset of the form $\{p[\omega]f^* : p \in L^*\}$, where $L^* \subseteq \Delta X$ is full-dimensional and $f^* = F^A(\sigma)$; and

(ii) $F^A(B^\varepsilon)$ contains a full-dimensional ball around $F^A(\sigma)$.

Proof.

(i) Fix a state ω and let $f^* = F^A(\sigma)$ and $f^*_{-B} := \sum_{s\in\sigma^{-B}} s_\omega c^s_\omega(A)$, where $\sigma^B := \{s \in \sigma : c^s(A) \in B\}$ and $\sigma^{-B} := \sigma\backslash\sigma^B$. Then $|\sigma^B| = N$. Without loss of generality, let B^ε denote an ε-neighborhood of σ^B. Let $\omega \in \Omega$ and note that for every $\sigma' \in B^\varepsilon$, there is a natural bijection between signals of σ and signals of σ'; specifically, $s \in \sigma$ and $s' \in \sigma'$ are related if and only if $c^s(A) = f^s = c^{s'}(A)$. For each $s \in \sigma$, let s' denote the corresponding signal in σ'.

Consider $\sigma' \in B^\varepsilon$ such that for all $s \in \sigma$ and all $\omega' \neq \omega$, $s_{\omega'} = s'_{\omega'}$. Thus, every such σ' induces an act of the form $p[\omega]f^*$, where

$$
\begin{aligned}
p &\in \left\{ \sum_{s'\in\sigma'} s'_\omega f^s_\omega + f^*_{-B} : s'_\omega \in (s_\omega - \varepsilon, s_\omega + \varepsilon) \text{ for all } s' \in \sigma', \text{ and } \sum_{s'\in\sigma'} s'_\omega = \sum_{s\in\sigma^B} s_\omega \right\} \\
&= \left\{ \sum_{s\in\sigma^B} (s_\omega + \delta^s) f^s_\omega + f^*_{-B} : |\delta^s| < \varepsilon \text{ and } \sum_{s\in\sigma^B} \delta^s = 0 \right\} \\
&= \left\{ f^*_\omega + \sum_{s\in\sigma^B} \delta^s f^s_\omega : |\delta^s| < \varepsilon \text{ and } \sum_{s\in\sigma^B} \delta^s = 0 \right\}
\end{aligned}
$$

So, it will suffice to show that the set

$$
C := \left\{ \sum_{s\in\sigma^B} \delta^s f^s_\omega : |\delta^s| < \varepsilon \text{ and } \sum_{s\in\sigma^B} \delta^s = 0 \right\}
$$

has dimension $N-1$ (clearly, C is convex). Note that $N-1$ is an upper bound on the dimension of C because C is a translation of a subset of ΔX.

Pick any $s^* \in \sigma^B$ and note that if $\sum_{s\in\sigma^B} \delta^s = 0$, then $\delta^{s^*} = -\sum_{s\in\sigma^B\backslash s^*} \delta^s$. Thus

$$
\begin{aligned}
C &= \left\{ \sum_{s\in\sigma^B\backslash s^*} \delta^s f^s_\omega - \sum_{s\in\sigma^B\backslash s^*} \delta^s f^{s^*}_\omega : |\delta^s| < \varepsilon \ \forall s \neq s^*, \text{ and } \left| \sum_{s\in\sigma^B\backslash s^*} \delta^s \right| < \varepsilon \right\} \\
&= \left\{ \sum_{s\in\sigma^B\backslash s^*} \delta^s (f^s_\omega - f^{s^*}_\omega) : |\delta^s| < \varepsilon \ \forall s \neq s^*, \text{ and } \left| \sum_{s\in\sigma^B\backslash s^*} \delta^s \right| < \varepsilon \right\}
\end{aligned}
$$

Let $\lambda^s := f^s_\omega - f^{s^*}_\omega$ for each $s \in \sigma^B\backslash s^*$. Then $\{\lambda^s : s \in \sigma^B\backslash s^*\}$ is linearly independent because $B_\omega = \{f^s_\omega : s \in \sigma^B\}$ is affinely independent. Let

$$
C' := \{0\} \cup \left\{ \frac{\varepsilon/2}{N-1} \lambda^s : s \in \sigma^B\backslash s^* \right\}
$$

Then C' is an affinely independent set of N vectors in $\mathbb{R}^N$, so that its convex hull has dimension $N-1$. Moreover, C contains the convex hull of C' because if $\lambda \in \text{co}(C')$, then there are scalars $\alpha^s \in [0,1]$ $(s \in \sigma^B)$ such that $\sum_{s\in\sigma^B} \alpha^s = 1$ and

$$\lambda = \alpha^{s^*} 0 + \sum_{s\in\sigma\setminus s^*} \alpha^s \frac{\varepsilon/2}{N-1} \lambda^s$$

We have $\lambda \in C$ because $\left|\frac{\alpha^s(\varepsilon/2)}{N-1}\right| < \varepsilon$ for all $s \in \sigma^B \setminus s^*$ and $\left|\sum_{s\in\sigma^B\setminus s^*} \frac{\alpha^s(\varepsilon/2)}{N-1}\right| < \varepsilon/2$. Since C contains the convex hull of C', and C' has dimension $N-1$, it follows that C has dimension $N-1$ (recall that $N-1$ is an upper bound on the dimension of C).

(ii) By part (i), $F^A(B^\varepsilon)$ (hence F^A) contains a subset of the form $L^*_\omega[\omega]f^*$ for each ω, where each set $L^*_\omega \subseteq \Delta X$ has full dimension. Since F^A is convex, apply Lemma 9 to get the result.

□

Lemma 11. *There is a full-dimensional set $L^* \subseteq \Delta X$ such that, for all ω, there exists an act h such that $L^*[\omega]h := \{p[\omega]h : p \in L^*\} \subseteq F^{A^*}$.*

Proof. Choose a lottery p^* in the interior of D (recall that D satisfies all requirements of Lemma 8). Fix a state ω. Then there is an A^*-interior experiment σ such that $F^{A^*}(\sigma) = p^*[\omega]h$ for some $h \in F$. By part (ii) of Lemma 10, F^{A^*} contains a full-dimensional ball around $p^*[\omega]h$. In particular, there is a convex, full-dimensional set $L^*_\omega \subseteq \Delta X$ such that p^* belongs to the interior of L^*_ω and $\{p[\omega]h : p \in L^*_\omega\} \subseteq F^{A^*}$. We may assume that $L^*_\omega \subseteq D$. Since $p^* \in D$, we can repeat this argument for all ω to get a family of convex, full-dimensional sets $L^*_\omega \subseteq \Delta X$, each containing p^* as an interior point, and acts $h^\omega \in F$ such that $\{p[\omega]h^\omega : p \in L^*_\omega\} \subseteq F^{A^*}$. Letting $L^* := \bigcap_{\omega\in\Omega} L^*_\omega$ completes the proof. □

Lemma 12. *Any linear representation $W^{A^*} : F^{A^*} \to \mathbb{R}$ of $\succsim^{A^*}$ on $\mathcal{E}^*(A^*)$ has a unique linear extension $W : F \to \mathbb{R}$. The extension represents a preference $\succsim$ on F satisfying all of the Anscombe-Aumann axioms except (possibly) the Non-Degeneracy axiom.*

Proof. A linear representation W^{A^*} exists by Step 1 of the proof (restrict V^{A^*} to the domain $\mathcal{E}^*(A^*)$ to form W^{A^*}). By Lemmas 9 and 11, F^{A^*} has full dimension, and therefore W^{A^*} has a unique linear extension $W : F \to \mathbb{R}$. This induces a complete and transitive relation $\succsim$ on F by letting $f \succsim g$ if and only if $W(f) \geq W(g)$. The Independence and Continuity axioms are satisfied by linearity of W.

To verify that $\succsim$ satisfies the State Independence axiom, suppose $p[\omega]h \succsim q[\omega]h$ and let $\omega' \in \Omega$ and $h' \in F$. We want to show that $p[\omega']h' \succsim q[\omega']h'$. By a standard result, there exist linear functions $U_\omega : \Delta X \to \mathbb{R}$ (unique up to positive affine transformation) such that $W(f) = \sum_\omega U_\omega(f_\omega)$ for all $f \in F$. Thus, $p[\omega]h \succsim q[\omega]h$ implies $U_\omega(p) \geq U_\omega(q)$.

Since $L^*[\omega]h^\omega \subseteq F^{A^*}$ for each ω, where $L^* \subseteq \Delta X$ is convex and has full dimension, there exists $r \in L^*$ and $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha)r \in L^*$ and $\alpha q + (1-\alpha)r \in L^*$. Thus, $(\alpha p+(1-\alpha)r)[\omega]h^\omega$, $(\alpha q+(1-\alpha)r)[\omega]h^\omega$, $(\alpha p+(1-\alpha)r)[\omega']h^{\omega'}$, and $(\alpha q+(1-\alpha)r)[\omega']h^{\omega'}$ are elements of F^{A^*}. Moreover, $(\alpha p + (1-\alpha)r)[\omega]h^\omega \succsim (\alpha q + (1-\alpha)r)[\omega]h^\omega$ because $W((\alpha p + (1-\alpha)r)[\omega]h^\omega) \geq W((\alpha q + (1-\alpha)r)[\omega]h^\omega)$ if and only if $U_\omega(p) \geq U_\omega(q)$ (recall that each U_ω is linear).

Since $\succsim^{A^*}$ satisfies State Independence (Axiom A5) on the domain $\mathcal{E}^*(A^*)$, it follows that $(\alpha p + (1-\alpha)r)[\omega']h^\omega \succsim (\alpha q + (1-\alpha)r)[\omega']h^\omega$. Therefore $U_{\omega'}(p) \geq U_{\omega'}(q)$, so that $U_{\omega'}(p) + \sum_{\hat\omega\neq\omega} U_{\hat\omega}(h'_{\hat\omega}) \geq U_{\omega'}(q) + \sum_{\hat\omega\neq\omega} U_{\hat\omega}(h'_{\hat\omega})$. Thus, $p[\omega']h' \succsim q[\omega']h'$, as desired. □

Note that, at this point, we cannot yet invoke the Anscombe-Aumann theorem to derive unique candidates for ν and v. This is because the Non-Degeneracy axiom only requires that *some* menu A (not necessarily A^*) has a non-degenerate preference $\succsim^A$. Step 3 of the proof will show that all preferences $\succsim^A$ (restricted to domains $\mathcal{E}^*(A)$) derive from the same, uniquely determined linear preference $\succsim$ on F. Then the Non-Degeneracy axiom will imply that $\succsim$ does not assign indifference among all acts, so that (combined with the above result proving that $\succsim$ satisfies State Independence) unique beliefs ν and preferences v exist by the Anscombe-Aumann theorem.

Step 3: Spreading the representation

Throughout the remainder of the proof, assume that DM2's utility index u has been normalized to take values in $[0,1]$.

A binary relation $\succsim$ on F is a *linear preference relation* if it has a linear representation; that is, a function $L : F \to \mathbb{R}$ such that $L(f) \geq L(g) \Leftrightarrow f \succsim g$, and $L(\alpha f + (1-\alpha)g) = \alpha L(f) + (1-\alpha)L(g)$ for all $f, g \in F$ and $\alpha \in [0,1]$. Recall that $F^A := \{F^A(\sigma) : \sigma \in \mathcal{E}^*(A)\} \subseteq F$ denotes the set of induced acts for menu A, and that F^A is convex because $F^A(\alpha\sigma \cup (1-\alpha)\sigma') = \alpha F^A(\sigma) + (1-\alpha)F^A(\sigma')$.

Definition 19. Let A and B be menus such that $\mathcal{E}^*(A)$ and $\mathcal{E}^*(B)$ are nonempty.

(i) A relation $\succsim$ on F *agrees with* $\succsim^A$ if, for all $\sigma, \sigma' \in \mathcal{E}^*(A)$, $\sigma \succsim^A \sigma' \Leftrightarrow F^A(\sigma) \succsim F^A(\sigma')$.

(ii) A *inherits* a representation from B if every linear preference relation $\succsim$ on F that agrees with B also agrees with A.

(iii) A and B *share a representation* if there is a unique linear preference relation $\succsim$ on F that agrees with both $\succsim^A$ and $\succsim^B$.

Lemma 13. *Let A and B be menus such that $\mathcal{E}^*(A)$ and $\mathcal{E}^*(B)$ are nonempty.*

(i) If $\dim(F^A) = \dim(F^A \cap F^B) \leq \dim(F^B)$*, then A inherits a representation from B.*

(ii) If $\dim(F^A) = \dim(F^A \cap F^B) = \dim(F^B) = \dim(F)$*, then A and B share a representation.*

Proof. By the Consistency axiom, $\succsim^A$ and $\succsim^B$ agree on the domain $F^A \cap F^B$. By Lemma 4, the restriction of V^B to $F^A \cap F^B$ is a linear function L. Since $F^A \cap F^B$ is convex and $\dim(F^A) = \dim(F^A \cap F^B) \leq \dim(F^B)$, L has a linear extension to F^A. Every such extension represents a linear preference relation $\succsim$ on F^A that agrees with A and B, proving (i). For (ii), note that L has a unique linear extension to F whenever $\dim(F^A \cap F^B) = \dim(F)$. $\square$

Lemma 14. *Suppose A inherits a representation from B. If $\succsim^B$ has an expected utility representation with parameters (v, ν), then so does $\succsim^A$.*

Proof. For convenience, let $Y = F^A$ and $Z = F^B$. Let $L^B : Z \to \mathbb{R}$ denote the expected utility representation for $\succsim^B$ and $L^A : Y \to \mathbb{R}$ a linear representation for $\succsim^A$ (such an L^A exists by Lemma 4 and the fact that F^A is convex). By the Consistency axiom, $\succsim^A$ and $\succsim^B$ induce the same linear ordering on $Y \cap Z$. Let $L^* : Y \cap Z \to \mathbb{R}$ denote the restriction of L^B to the domain $Y \cap Z$. Since $\dim(Y \cap Z) = \dim(Y)$ and $Y \cap Z$ is convex, L^* has a unique linear extension to Y. Thus, we may assume $L^A(f) = L^*(f)$ for all $f \in Y \cap Z$. So, on $Y \cap Z$, L^A takes the desired form.

Let $f \in Y \backslash Z$. Since Y is convex and $\dim(Y) = \dim(Y \cap Z)$, there are $g, h \in Y \cap Z$ and $\alpha \in (0, 1)$ such that $h = \alpha f + (1 - \alpha)g$. Then $L^A(h) = \alpha L^A(f) + (1 - \alpha)L^A(g)$. Since $L^A = L^*$ on $Y \cap Z$, it follows that

$$
\begin{aligned}
L^A(f) &= \frac{1}{\alpha}[L^*(h) - (1 - \alpha)L^*(g)] \\
&= \frac{1}{\alpha}\left[\sum_\omega v(h_\omega)\nu_\omega - (1 - \alpha)\sum_\omega v(g_\omega)\nu_\omega\right] \\
&= \frac{1}{\alpha}\sum_\omega \left[v(\alpha f_\omega + (1 - \alpha)g_\omega) - (1 - \alpha)v(g_\omega)\right]\nu_\omega \\
&= \frac{1}{\alpha}\sum_\omega \left[\alpha v(f_\omega) + (1 - \alpha)v(g_\omega) - (1 - \alpha)v(g_\omega)\right]\nu_\omega \\
&= \sum_\omega v(f_\omega)\nu_\omega,
\end{aligned}
$$

as desired. $\square$

Definition 20. Let A be a menu.

1. If $f \in A$, the *support* of f is the set $S^A(f) := \{s \in S : c^s(A) = f\}$.

2. A is a *k-menu* if $|A| = k \geq 2$ and each $f \in A$ has nonempty support.

3. A is *independent* if it is a k-menu for some k and, for each ω, there is an N-menu $B \subseteq A$ such that $B_\omega := \{f_\omega : f \in B\}$ is affinely independent.

Lemma 15. *Suppose A is a k-menu.*

(i) If $f \in A$, then $S^A(f)$ is a convex cone and has full dimension (in S)

(ii) There exists an A-interior experiment σ

(iii) If A is independent, then F^A has full dimension (in F)

Proof.

(i) First, observe that $s \in S^A(f)$ if and only if, for all $g \in A$,

$$\sum_\omega \frac{s_\omega \mu_\omega}{\sum_{\omega'} s_{\omega'}\mu_{\omega'}} u(f_\omega) > \sum_\omega \frac{s_\omega \mu_\omega}{\sum_{\omega'} s_{\omega'}\mu_{\omega'}} u(g_\omega)$$
$$\Leftrightarrow \sum_\omega s_\omega \mu_\omega u(f_\omega) > \sum_\omega s_\omega \mu_\omega u(g_\omega)$$

It is now straightforward to verify that if $s, t \in S^A(f)$, then $\lambda s \in S^A(f)$ for all $\lambda > 0$ such that $\lambda s \in S$, and $\alpha s + (1-\alpha)t \in S^A(f)$ for all $\alpha \in [0,1]$. Thus, $S^A(f)$ is a convex cone. To see that it is a full-dimensional subset of $S := [0,1]^\Omega \backslash 0$, note that since the above inequalities are strict, there is an open ball (in the subspace topology for S derived from the standard topology on $\mathbb{R}^\Omega$) around each $s \in S^A(f)$ that preserves the inequality; since the open ball has full dimension, the result follows.

(ii) Since A is finite and each set $S^A(f)$ is a convex cone, there are signals s^f $(f \in A)$ such that $c^{s^f}(A) = f$ and, for each ω, $\sum_{f \in A} s^f_\omega \leq 1$ (simply choose any signals $s^f \in S^A(f)$ and, if necessary, scale them all down by a factor $\alpha \in (0,1)$ to ensure $\sum_{f \in A} s^f_\omega \leq 1$). For each ω, there is an $f \in A$ such that $u(f_\omega) \geq u(g_\omega)$ for all $g \in A$. Thus, s^f_ω can be increased as needed to ensure $\sum_{f \in A} s^f_\omega = 1$. Repeat this for each ω to get a well-defined experiment $\sigma = \{s^f : f \in A\}$.

(iii) By part (ii), there is an A-interior σ and, hence, a ε-neighborhood around σ. Let $\omega \in \Omega$. Since A is independent, there is an N-menu $B \subseteq A$ such that $B_\omega = \{f_\omega : f \in B\}$ is affinely independent. Now apply Lemma 10.

□

Definition 21. A finite, nonempty set $\mathcal{C}$ of convex cones in S is a *conic decomposition* if $\mathcal{C} = \left\{S^A(f) : f \in A\right\}$ for some k-menu A. For each k-menu A, the set

$$\mathcal{C}(A) := \left\{S^A(f) : f \in A\right\}$$

is the *conic decomposition* for A.

Definition 22. For each k-menu A and $f \in A$, let $U(f) := (\mu_\omega u(f_\omega))_{\omega\in\Omega}$ denote the *(virtual) utility coordinate* for f, and let $U(A) := \{U(f) : f \in A\}$ denote the *utility profile for* A. If a set $U \subseteq \mathbb{R}^\Omega_+$ satisfies $U = U(A)$ for some k-menu A, then U is a k*-utility profile*. Finally, a finite set $U \subseteq \mathbb{R}^\Omega_+$ is a *utility profile* if U is a k-utility profile for some k.

Lemma 16. *If A and B are k-menus such that $U(A) = U(B)$, then $\mathcal{C}(A) = \mathcal{C}(B)$.*

Proof. This follows immediately from the definition of $U(A)$ and the fact that $s \in S^A(f)$ if and only if $\sum_\omega s_\omega\mu_\omega u(f_\omega) > \sum_\omega s_\omega\mu_\omega u(g_\omega)$ for all $g \in A\backslash\{f\}$. □

By Lemma 16, each utility profile U has an associated conic decomposition $\mathcal{C}(U)$. Specifically, $\mathcal{C}(U)$ is the unique $\mathcal{C}$ such that $U(A) = U$ implies $\mathcal{C}(A) = \mathcal{C}$.

Definition 23. Let U be a utility profile and $z = (z_\omega)_{\omega\in\Omega} \in U$. The *support* of z in U is the set

$$S^U(z) := \left\{s \in S : \forall z' \in U,\ \sum_\omega s_\omega z_\omega > \sum_\omega s_\omega z'_\omega\right\} \tag{9}$$

Definition 24. Let U be a utility profile. For each $z \in U$ and $s \in S^U(z)$, let $H(z, s) := \{\lambda \in \mathbb{R}^\Omega : s \cdot (\lambda - z) \leq 0\}$. The *support polytope of z in U*, denoted $T(z, U)$, is defined as

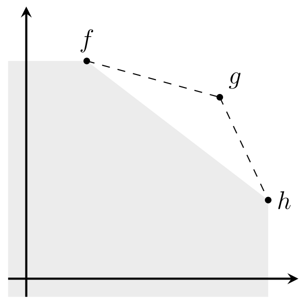
$$T(z, U) := \bigcap_{s\in S^U(z)} H(z, s). \tag{10}$$

The *polytope of U*, denoted $T(U)$, is given by

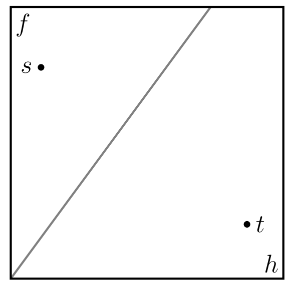
$$T(U) := \bigcap_{z\in U} T(z, U). \tag{11}$$

A polytope $T \subseteq \mathbb{R}^\Omega$ is a *decision polytope* if $T = T(U)$ for some utility profile U; it is a k*-polytope* if $T = T(U(A))$ for some k-menu A.

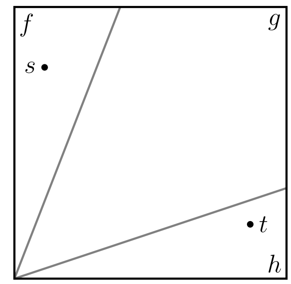
Definition 25. Let T be a decision polytope. For each face F of T, let $\eta^F \in S^\Omega_+ := \{\eta \in \mathbb{R}^\Omega_+ : \|\eta\| = 1\}$ such that η^F is normal to the hyperplane associated with F. Let $\mathcal{N}(T) := \{\eta^F : F \text{ is a face of } T\}$ denote the set of *normals* for T.



(a) Utility Space



(b) Signal space for $A = \{f, h\}$



(c) Signal space for $B = \{f, g, h\}$

Figure 11: Illustration of Lemma 17. The shaded region in (a) is $T(A)$. Experiment $\sigma = [s, t]$ is constructed so that f is chosen at s and h is chosen at t. If the (utility coordinate) of g is close to the face joining f and h, then $c^s(B) = f$ and $c^t(B) = h$ as well, where $B = \{f, g, h\}$. How close g needs to be to the face depends on s and t (the dashed lines in (a) are perpendicular to the gray lines in (c)). Thus, $F^A(\sigma') = F^B(\sigma')$ for all σ' in a neighborhood of σ, so that A inherits a representation from B.

Figure 1 in the main text illustrates the relationship between a menu A, its utility profile $U(A)$, and the associated decision polytope and conic decomposition. Each act in a menu A determines a coordinate $U(f)$ in utility space, and the set $U(A)$ of utility coordinates yields a polytope $T(U(A))$ in utility space (in the figure, the shaded region is the polytope). An act is chosen under some signal if and only if $U(f)$ is an extreme point of the polytope. For any such act f, the set of signals s where $c^s(A) = f$ is a cone in S. Faces of the polytope correspond to signals making DM2 indifferent between two or more acts in A. Thus, any s perpendicular to a face of the polytope lies on a hyperplane in signal space separating the cones corresponding to two or more acts.

The next task is to show that every k-menu inherits a representation from some independent ℓ-menu, and that all independent menus share a representation. The proof is divided into three parts.

Part 1: Every k-menu inherits a representation from some independent menu

Lemma 17 (Vertex Expansion)**.** *Let A be a k-menu. There is an act $g \notin A$ such that $B = A \cup \{g\}$ is a $(k+1)$-menu and A inherits a representation from B.*

Proof. Let $\sigma \in \mathcal{E}$ be A-interior and choose $2\varepsilon > 0$ such that $B^{2\varepsilon}$ is a 2ε-neighborhood of σ. Then B^{ε} is an ε-neighborhood where, for all $\sigma' \in B^{\varepsilon}$ and all $s \in \sigma'$, the closure of $Q^{s,\varepsilon}$ is in the interior of $S^A(f)$, where $f = c^s(A)$.

Let $f \in A$. For each $\sigma' \in B^{\varepsilon}$ and each $s \in \sigma$, consider the half-space $H(f, s) := \{\lambda \in \mathbb{R}^{\Omega}_+ : s \cdot (\lambda - U(f)) \leq 0\}$. This is the half-space (containing the origin) where the bounding hyperplane has normal s and passes through $U(f)$. Let T^* be (the closure of) the intersection over all $H(f,s)$ where $f \in A$ and $s \in \sigma' \in B^\varepsilon$. Notice that for each f, the set $B^\varepsilon(f) := \{s \in S : c^s(A) = f \text{ and } s \in \sigma' \in B^\varepsilon\}$ is an (open) convex cone in S, and a strict subset of $\text{int}(S^A(f))$ by our choice of ε. Thus, $B^\varepsilon(f)$ and $B^\varepsilon(f')$ are strictly separated whenever $f \neq f'$, and therefore $T(A) \subsetneq T^*$. Pick any point $u^* \in [T^* \backslash T(A)] \cap \mathbb{R}^\Omega_+$ and let $g \in F$ such that $U(g) = u^*$. Then $B = A \cup \{g\}$ is the desired menu.

To see why A and B share a representation, note that (by construction) $c^s(A) = c^s(B)$ for all $s \in \sigma' \in B^\varepsilon$. Hence, $F^A(\sigma') = F^B(\sigma')$ whenever $\sigma' \in B^\varepsilon$. Since $\dim(F^A) = \dim(F^A(B^\varepsilon))$ and $F^A(B^\varepsilon) = F^B(B^\varepsilon) \subseteq F^B$, it follows that $\succsim^A$ inherits a representation from $\succsim^B$. □

Lemma 18. *Let A be a k-menu. There exists an independent menu B such that A inherits a representation from B.*

Proof. Fix an A-interior experiment σ and a neighborhood B^ε of the form used in the proof of Lemma 17. It is easy to see that a similar argument can be used to add N additional vertices to the region $T^* \backslash T(A)$ to yield a $(k+N)$-polytope. Moreover, these vertices can be chosen so that for each state ω, the ω coordinates yield N distinct, interior utility values. We are free to pick any N lotteries $p^1_\omega, \ldots, p^N_\omega$ yielding these utility values. Clearly, these can be chosen to form an affinely independent set. Now let $f^i = (p^i_\omega)_{\omega \in \Omega} \in F$, and let $B = A \cup \{f^1, \ldots, f^N\}$. □

Part 2: Oriented translations share a representation

Definition 26. Let A and B be independent menus. Then B is a *translation* of A if there exists $\lambda^* \in \mathbb{R}^\Omega$ such that $T(B) = T(A) + \lambda^* := \{\lambda + \lambda^* : \lambda \in T(A)\}$. The notation $B = A + \lambda^*$ means $T(B) = T(A) + \lambda^*$.

Lemma 19. *If $B = A + \lambda^*$, then:*

(i) The map $\psi : U(A) \to U(B)$ given by $\psi(z) := z + \lambda^*$ is a bijection. Hence, there is a bijection $\psi : A \to B$ where $\psi(f)$ denotes the unique $g \in B$ such that $U(g) = U(f) + \lambda^*$.

(ii) $\mathcal{C}(B) = \mathcal{C}(A)$.

Proof. Part (i) is clear. For part (ii), observe that $s \in S^A(f)$ if and only if

$$\begin{aligned}
&\sum_\omega s_\omega u(f_\omega) > \sum_\omega s_\omega u(g_\omega) \\
\Leftrightarrow &\sum_\omega s_\omega [\mu_\omega u(f_\omega) + \lambda^*_\omega] > \sum_\omega s_\omega [\mu_\omega u(g_\omega) + \lambda^*_\omega] \\
\Leftrightarrow & s \in S^B(\psi(f))
\end{aligned}$$

It follows that $\mathcal{C}(B) = \mathcal{C}(A)$. $\square$

Definition 27. Suppose B is a translation of A, and let $\psi : A \to B$ denote the associated bijection (Lemma 19). The *affine path from f to $\psi(f)$* is the map $\alpha \mapsto f^\alpha := (1-\alpha)f + \alpha\psi(f)$ for $\alpha \in [0,1]$ and the *affine path from A to B* is the map $\alpha \mapsto A^\alpha := \{f^\alpha : f \in A\}$ for $\alpha \in [0,1]$.

Definition 28. A bijection $\varphi : P \to Q$ between to sets of N lotteries is *oriented* if

(i) For all $p, p' \in P$, $u(p) > u(p')$ implies $u(\varphi(p)) > u(\varphi(p'))$, and

(ii) For each $\alpha \in [0,1]$, the set $\{(1-\alpha)p + \alpha\varphi(p) : p \in P\}$ is affinely independent.

Independent menus A and B are *oriented* if B is a translation of A and, for each ω, the map $\varphi_\omega : A_\omega \to B_\omega$ given by $\varphi_\omega(f_\omega) := \psi(f)_\omega$ is oriented, where $A_\omega := \{f_\omega : f \in A\}$, $B_\omega := \{g_\omega : g \in B\}$, and $\psi : A \to B$ is the associated bijection (Lemma 19).

Figure 3 in the main text illustrates the concept of orientedness. Note that not all translations $B = A + \lambda^*$ are oriented; in fact, it is possible to construct menus A and B such that $U(A) = U(B)$ (so that B is trivially a translation of A) but where A and B are not oriented, so that even $T(A) = T(B)$ is not enough to guarantee that A and B are oriented. So, some care is needed when applying the following lemma:

Lemma 20. *If A and B are oriented menus, then A and B share a representation.*

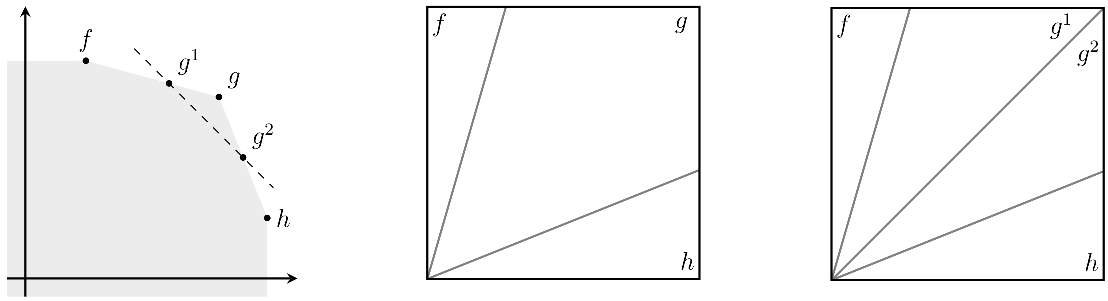
Proof. Since A and B are oriented, there is a $\lambda^* \in \mathbb{R}^\Omega$ such that $B = A + \lambda^*$ and an associated bijection $\psi : A \to B$ (Lemma 19).

Consider the affine path associated with ψ (Definition 27), and note that for each α, $A^\alpha = A + \alpha\lambda^*$; that is, $T(A^*) = T(A) + \alpha\lambda^*$.

Thus, every A-interior (B-interior) experiment σ is also A^α-interior. Pick such a σ and a corresponding neighborhood B^ε, and let $f^\alpha := F^{A^\alpha}(\sigma)$. Importantly, $F^{A^\alpha}(B^\varepsilon)$ contains a full-dimensional subset of F because A^α is an independent menu (since A and B are oriented).

For every α, f^α is in the interior of $F^{A^\alpha}(B^\varepsilon)$. Let $\delta(\alpha) > 0$ denote the radius of the largest open ball around f^α contained in $F^{A^\alpha}(B^\varepsilon)$; call this ball B^α. Clearly, f^α and $\delta(\alpha)$ are continuous in α. Therefore $\delta^* = \min_\alpha \delta(\alpha)$ is well-defined.

Now construct a finite sequence $\alpha(0), \alpha(1), \ldots, \alpha(I)$ such that $\alpha(0) = 0$, $\alpha(I) = 1$, and $d(f^{\alpha(i)}, f^{\alpha(i-1)}) < \delta^*/2$ for all $i = 1, \ldots, I$, where d denotes the standard Euclidean metric. This can be done because f^α is continuous in α. Notice that $f^{\alpha(i)} \in B^{\alpha(i-1)}$ for all $i = 1, \ldots, I$. Thus, $B^{\alpha(i)}$ and $B^{\alpha(i-1)}$ intersect in a full-dimensional region, so that $A^{\alpha(i)}$ and $A^{\alpha(i-1)}$ share a representation. Hence, A and B share a representation. $\square$



(a) Utility Space (b) Signal space for $A = \{f, g, h\}$ (c) Signal space for $B = \{f, g^1, g^2, h\}$

Figure 12: Illustration of Lemma 21. The shaded region in (a) is $T(A)$. T' is formed by clipping off the region above the dashed line, effectively replacing coordinate g with coordinates g^1 and g^2. The region in signal space where g is chosen from $A = \{f, g, h\}$ is divided into regions for g^1 and g^2 in menu $B = \{f, g^1, g^2, h\}$ (in this example, the dashed line is orthogonal to e). If the acts yielding utility coordinates g^1 and g^2 are sufficiently close to g, then A-interior experiments σ yield induced acts $F^A(\sigma)$ and $F^B(\sigma)$ that are close to each other.

Part 3: Independent menus share a representation

Lemma 21 (Face Expansion)**.** *Let A be an independent menu and suppose $\mathcal{N} = \mathcal{N}(A) \cup \{\lambda\}$ for some $\lambda \in S^\Omega_+$. Then there is an independent menu B such that:*

(i) $\mathcal{N}(B) = \mathcal{N}$

(ii) A and B share a representation.

Proof. Fix an A-interior experiment σ and an ε-neighborhood B^ε around σ. Without loss of generality, no $s \in \sigma$ is of the form $s = \gamma\lambda$ for any $\gamma > 0$ (if necessary, choose some other $\sigma' \in B^\varepsilon$ and redefine σ to be σ'). Let $f^* := F^A(\sigma)$. Since A is independent, the set $\{F^A(\sigma') : \sigma' \in B^\varepsilon\}$ contains a ball of radius δ around f^* for some $\delta > 0$.

Let $H := \{\lambda' \in \mathbb{R}^\Omega : \lambda \cdot \lambda' = \zeta\}$ denote the (unique) hyperplane with normal λ that intersects the boundary (but not the interior) of $T(A)$. The half-space $H^*(\zeta) := \{\lambda' \in \mathbb{R}^\Omega : \lambda \cdot \lambda' \leq \zeta\}$ below H contains $T(A)$. Shifting H^* toward the origin by a small amount (that is, taking $H^*(\zeta')$ with $\zeta' < \zeta$) and intersecting with $T(A)$ yields a new decision polytope T' where one or more vertices of $T(A)$ are split into multiple vertices. This means that for at least one $f \in A$, the vertex $z^f = U(f) \in T(A)$ is split into vertices $z^{f_1}, \ldots, z^{f^n}$ in T', and the set $S^A(f)$ is divided into convex cones $S(f^i) \subseteq S^A(f)$ where $S(f^i) := \{s \in S : s \cdot z^{f^i} > s \cdot z \; \forall z' \neq z^{f^i}\}$.

By construction, T' has a face with normal λ. By letting $\zeta' \to \zeta$, T' converges to $T(A)$ (in the Hausdorff metric). Thus, if the vertex $z^f \in T(A)$ corresponding to some $f \in A$ is split into $z^{f_1}, \ldots, z^{f^n}$ in T', the coordinates z^{f^i} each converge to z^f as $\zeta' \to \zeta$. Therefore, acts f^i such that $U(f^i) = z^{f^i}$ can be chosen such that $f^i \to f$ as $\zeta' \to \zeta$. Moreover, the acts corresponding to new vertices can be chosen so that the resulting menu B is independent (perturb the constituent lotteries along indifference curves for u if necessary).

Thus, there is a ζ' near ζ for which the corresponding menu B satisfies $d(f^*, F^B(\sigma)) < \delta$; that is, $F^B(\sigma)$ is in the interior of the ball of radius δ around f^*. Since B is independent, F^B contains a ball of radius δ' around $F^B(\sigma)$ for some $\delta' > 0$. Thus, $\dim(F^A \cap F^B) = \dim(F)$, so that A and B share a representation. $\square$

Lemma 22. *Suppose A is a k-menu and $B \subseteq A$ such that $c^e(A) \in B$. There exists an experiment σ such that $c^s(A) \in B$ for all $s \in \sigma$. Moreover, σ may be chosen so that for each $f \in B$, σ contains a signal s^f such that $c^{s^f}(A) = B$.*

Proof. Let $f^e \in B$ denote the act satisfying $c^e(A) \in B$. For each $f \in B \backslash f^e$, pick s^f such that $c^{s^f}(A) = f$; such s^f exist because A is a k-menu. Let $s := \sum_{f \in B \backslash f^e} s^f$, and choose $\alpha \in (0,1)$ such that $e - \alpha s \in S^A(f^e)$. Such an α exists because for small enough α, $e - \alpha s$ is close to $e \in S^A(f^e)$, which is a full-dimensional subset of $S = [0,1]^\Omega \backslash 0$. Finally, let $\sigma = \{\alpha s^f : f \in B \backslash f^e\} \cup \{e - \alpha s\}$. Since $c^{\lambda t} = c^t$ for all $\lambda > 0$ such that $\lambda t \in S$, it follows that σ is a well-defined experiment satisfying all desired properties. $\square$

Lemma 23. *Suppose U is a k-utility profile and U' is an ℓ-utility profile such that $T = T(U)$ and $T' = T(U')$ satisfy $\frac{1}{W} e \in \mathcal{N}(T) \cap \mathcal{N}(T')$. For each choice of A and B such that $U = U(A)$ and $U' = U(B)$, there exists an N-utility profile U^* and a $\lambda \in \mathbb{R}^\Omega$ such that:*

(i) $U \cup U^*$ is a $(k+N)$-utility profile and $U' \cup (U^* + \lambda)$ is a $(\ell + N)$-utility profile

(ii) There is a $z \in U^*$ such that $e \in S^{U \cup U^*}(z)$ and $e \in S^{U' \cup (U^* + \lambda)}(z + \lambda)$

(iii) If $U^* = U(A^*)$ and $U^* + \lambda = U(B^*)$, then A inherits a representation from $A \cup A^*$ and B inherits a representation from $B \cup B^*$.

Proof. Let A and B satisfy $U = U(A)$ and $U' = U(B)$. Choose an A-interior experiment σ and a corresponding neighborhood B^ε, and a B-interior σ' with neighborhood $B^{\varepsilon'}$. As in the proof of Lemma 17, the half-spaces corresponding to signals $s \in \hat{\sigma} \in B^\varepsilon$ passing through the point $U(f^s)$ (where $f^s = c^s(A)$) intersect to form a space $T^*(A)$ such that $T(A) \subseteq T^*(A)$. Moreover, $T^*(A) \backslash T(A)$ contains a full-dimensional subset of $\mathbb{R}^\Omega$ near the face of $T(A)$ with normal e because every $s \in \hat{\sigma} \in B^\varepsilon$ is bounded away from e. In other words, $T^*(A) \backslash T(A)$ contains a full-dimensional subset of the region above the hyperplane corresponding to this face. A similar argument yields a region $T^*(B)$ for which analogous statements hold.

Thus, there is a $\delta > 0$ such that both $T^*(A) \backslash T(A)$ and $T^*(B) \backslash T(B)$ contain an open ball of radius δ. Letting D^A denote such a ball in $T^*(A) \backslash T(A)$ and D^B the ball in $T^*(B) \backslash T(B)$, it follows that $D^B = D^A + \lambda$ for some $\lambda \in \mathbb{R}^\Omega$.

The profile U^* is constructed as follows. First, pick a point $z^1 \in D^A$. Then $z^1 + \lambda \in D^B$. By our choice of D^A and D^B, we have that $T(U \cup \{z^1\})$ is a $(k+1)$-polytope such that $e \in S^{U \cup \{z^1\}}(z^1)$; that is, if some act f^1 satisfies $U(f^1) = z^1$, then $c^e(A \cup \{f^1\}) = f^1$. Since this is a strict preference, there is in fact a full-dimensional, convex set of signals s such that $c^s(A \cup \{f^1\}) = f^1$, and e belongs to the interior of this set. Similar statements hold for $B \cup \{g^1\}$ for any g^1 such that $U(g^1) = z + \lambda$. Therefore, there is a full-dimension set of signals s such that $c^s(A \cup \{f^1\}) = f^1$ and $c^s(B \cup \{g^1\}) = g^1$. Call the set of all such s the *support* of z^1.

We now proceed by induction. Suppose $U^* = \{z^1, \ldots, z^n\} \subseteq D^A$ such that each $z \in U^*$ has full-dimensional support. That is, for any A^* such that $U(A^*) = U^*$ and each $f \in A^*$, the set $S^z = S^{A \cup A^*}(f) \cap S^{B \cup (A^* + \lambda)}(g)$ has full dimension, where $g \in B^*$ satisfies $U(g) = U(f) + \lambda$. Pick an s in the interior of S^z such that $s^z \neq e$ and consider the hyperplane $H(s; z)$ with normal s passing through z. Now pick a point $z^{n+1} \in H(s; z) \backslash z$; if z^{n+1} is sufficiently close to z, then $z^{n+1} \in D^A$, $T(U \cup U^* \cup \{z^{n+1}\})$ is a $(k+n+1)$-polytope, and $T(U' \cup (U^* \cup \{z^{n+1}\} + \lambda))$ is an $(\ell + n + 1)$-polytope. Moreover, z^{n+1} has full dimensional support.

The resulting set $U^* = \{z^1, \ldots, z^N\}$ clearly satisfies (i) and (ii). For (iii), note that our original choice of D^A and D^B guarantees that for all $s \in \hat{\sigma} \in B^\varepsilon$, $c^s(A \cup A^*) = c^s(A)$ and $s' \in \hat{\sigma}' \in B^{\varepsilon'}$ implies $c^{s'}(B \cup B^*) = c^{s'}(B)$. Thus, $F^A(B^\varepsilon) \subseteq F^{A \cup A^*}$ and $F^B(B^\varepsilon) \subseteq F^{B \cup B^*}$, so that $\dim(F^A) \leq \dim(F^{A \cup A^*})$ and $\dim(F^B) \leq \dim(F^{B \cup B^*})$. □

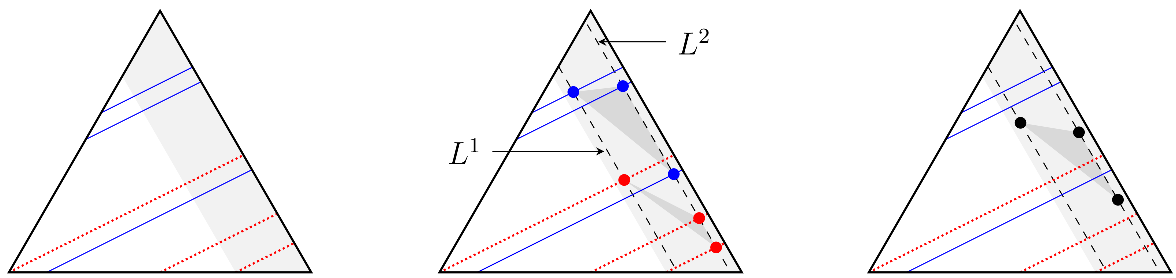
Lemma 24. *Suppose $U, U' \subseteq (0,1)$ are sets of cardinality N. There exist sets $P, Q \subseteq \Delta X$ and a bijection $\varphi : P \to Q$ such that*

(i) $U = \{u(p) : p \in P\}$ and $U' = \{u(q) : q \in Q\}$, and

(ii) φ is oriented.

Proof. Figure 13 illustrates the idea of the proof. Consider the indifference curves (hyperplanes) in ΔX corresponding to the utilities in $U \cup U'$. There is an edge E of ΔX such that each of these planes intersects the (relative) interior of E. Specifically, E is any edge connecting lotteries δ_b and δ_w for any choice of $b, w \in X$ such that $u(b) \geq u(x) \geq u(w)$ for all $x \in X$. Since each utility level is interior, it can be expressed as a non-degenerate mixture of $u(b)$ and $u(w)$, forcing the associated hyperplane to intersect the relative interior of E. Parallel to this edge is an interior line L passing through (the interior of) each hyperplane, so that in fact there is an $\varepsilon > 0$ such that every parallel ε perturbation of L passes through each hyperplane. Let $B \subseteq \Delta X$ denote the region spanned by these perturbations; clearly, B has dimension equal to that of ΔX (namely, $N - 1$).

Now pick $N - 1$ lines $L^1, \ldots, L^{N-1}$ in B, each parallel to L, such that the convex hull of $\{L^1, \ldots, L^{N-1}\}$ has dimension $N - 1$. Rank the numbers in $u^i \in U$ so that $u^1 > u^2 > \ldots >$



(a) Utility Levels (b) P (blue dots) and Q (red dots) (c) P^α, $\alpha = 0.5$

Figure 13: Illustration of Lemma 24. The solid (blue) lines are the utility levels for U, and the dotted (red) lines are utility levels for U'. The shaded region in (a) is the region $B \subseteq \Delta X$ referenced in the proof. With this construction, every set P^α is affinely independent.

u^N. For $i = 1, \ldots, N-1$, let p^i be the (unique) intersection of L^i and the indifference plane for utility u^i, and let p^N be the unique intersection of L^{N-1} with the indifference plane for utility u^N. Observe that $\{p^1, \ldots, p^{N-1}\}$ lie on a hyperplane H in ΔX and that p^N is not in the affine hull of H because L^{N-1} passes through H at a single point (p^{N-1}) while p^N lies at a different point on L^{N-1}. Thus, $P = \{p^1, \ldots, p^N\}$ is affinely independent.

Using the same lines $L^1, \ldots, L^{N-1}$ and the same rank-based construction for U' yields an affinely independent set $Q = \{q^1, \ldots, q^N\}$ where $u(q^1) > \ldots > u(q^N)$.

Now consider $P^\alpha := \{(1-\alpha)p^i + \alpha q^i : i = 1, \ldots, N\}$. Observe that $(1-\alpha)u(p^i) + \alpha u(q^i) > (1-\alpha)u(p^{i+1}) + \alpha u(q^{i+1})$ for all $i = 1, \ldots, N-1$ because $u(p^i) > u(p^{i+1})$ and $u(q^i) > u(q^{i+1})$. Notice also that $(1-\alpha)p^i + \alpha q^i$ is on line L^i ($i = 1, \ldots, N-1$) and $(1-\alpha)p^N + \alpha q^N$ is on L^{N-1}. Thus, by the same argument, P^α is affinely independent. Hence, the map $\varphi : P \to Q$ given by $\varphi(p^i) = q^i$ ($i = 1, \ldots, N$) is oriented. □

Lemma 25. *If A and B are independent, then A and B share a representation.*

Proof. By Lemma 21, we may assume that $e \in \mathcal{N}(A)$ and $e \in \mathcal{N}(B)$. Then, by Lemma 23, there is a utility profile U and a $\lambda \in \mathbb{R}^\Omega$ such that if $U = U(A^*)$ and $U' := U + \lambda = U(B^*)$, then A and $A' := A \cup A^*$ share a representation, and B and $B' := B \cup B^*$ share a representation. In fact, by Lemma 22, A' shares a representation with A^* provided A^* is independent. Similarly, B' shares a representation with B^* provided B^* is independent. Therefore, it will suffice to find independent menus A^* and B^* such that $U = U(A^*)$, $U' = U(B^*)$, and such that A^* and B^* share a representation.

To do so, choose a state ω and apply Lemma 24 to the sets $U_\omega := \{z_\omega : z \in U\}$ and $U'_\omega := \{z'_\omega : z' \in U'\}$ to get affinely independent sets $P_\omega := \{p^z_\omega : z \in U\}$ and $Q_\omega := \{q^{z'}_\omega : z' \in U'\}$ such that $u(p^z_\omega) = z_\omega$ and $u(q^{z'}_\omega) = z'_\omega$ for all $z \in U$ and $z' \in U'$ (if necessary, apply a small perturbation to U and U' in order to get N distinct utility values in U_ω for each ω, and N

distinct utility values in U'_ω for all ω). Repeating this for each ω yields acts $f^z := (p^z_\omega)_{\omega\in\Omega}$ and $g^{z'} := (q^{z'}_\omega)_{\omega\in\Omega}$ for each $z \in U$ and $z' \in U'$. Then $A^* := \{f^z : z \in U\}$ and $B^* := \{g^{z'} : z' \in U'\}$ are oriented, so that by Lemma 20, A^* and B^* share a representation. □

Lemma 26. *There is a unique, linear $L^* : F \to \mathbb{R}$ such that, for all k-menus A, $V^A(\sigma) := L^*(F^A(\sigma))$ represents $\succsim^A$ for all $\sigma \in \mathcal{E}^*(A)$.*

Proof. By Lemma 25, all independent menus share a representation. This means there is a unique linear $\succsim$ on F that agrees with each relation $\succsim^A$. This $\succsim$ also agrees with $\succsim^A$ since every k-menu inherits a representation from an independent menu (Lemma 18). To construct L^*, choose any independent menu A and consider the linear representation V^A (Lemma 4) restricted to the domain F^A. Since F^A has full dimension, V^A has a unique linear extension to F. Take L^* to be this extension. □

Proof of Theorem 1

Let A be an arbitrary menu and consider the utility profile $U(A)$. If $U(A)$ consists of a single point, or if $\mathcal{E}^*(A) = \emptyset$, there is nothing to prove. Otherwise, let $\sigma, \sigma' \in \mathcal{E}^*(A)$. Then there is a submenu $A' \subseteq A$ that is a k-menu (for some k) such that $F^A(\sigma) = F^{A'}(\sigma)$ and $F^A(\sigma') = F^{A'}(\sigma')$. By the Consistency Axiom (A7), $\sigma \succsim^A \sigma'$ if and only if $\sigma \succsim^{A'} \sigma'$. Hence, any linear representation for $\mathcal{E}^*(A')$ gives a linear representation for $\mathcal{E}^*(A)$. In particular, if some pair (ν, v) gives an expected utility representation on some (any) independent menu A, then it gives a linear representation for $\mathcal{E}^*(B)$ for all menus B (Lemma 14).

The only remaining task is to pin down the desired uniqueness properties for ν and v. First, note that by the Non-Degeneracy axiom, the (unique) linear representation L^* of Lemma 26 must be non-constant; otherwise, by the previous paragraph, every $\succsim^A$ assigns indifference among all experiments in $\mathcal{E}^*(A)$. Thus, by Lemma 12, $\succsim^{A^*}$ (uniquely) extends to $\succsim$ on F (where A^* is the symmetric menu constructed in Step 2), so that $\succsim$ satisfies all of the Anscombe-Aumann axioms, including Non-Degeneracy. Thus, $\succsim$ has an expected utility representation with a unique ν and a unique (up to positive affine transformation) utility index v. Since L^* is a linear representation for $\succsim$, it follows that the expected utility representation holds for all menus $\succsim^A$ on $\mathcal{E}^*(A)$.

B Proof of Theorem 2

Axioms B1–B5 imply that for each s, c^s is rationalized by an Anscombe-Aumann representation with prior μ^s and utility index u^s. Axiom B4 implies that u^s is a positive affine transformation of $u^{s'}$ for all s, s', so we may assume $u^s = u$ for all s. That is, every $\succsim^s$ has a representation of the form

$$f \succsim^s g \Leftrightarrow \sum_{\omega} \mu^s_\omega u(f_\omega) \geq \sum_{\omega} \mu^s_\omega u(g_\omega)$$

We will refer to this as the *expected utility representation for* $\succsim^s$. To complete the proof, we demonstrate existence of a (full support) μ such that, for all s, μ^s is the Bayesian posterior induced by prior μ and signal s.

Lemma 27. *If $s_\omega > 0$, then $\mu^s_\omega > 0$.*

Proof. If $\mu^s_\omega = 0$, then

$$u(p)\mu^s_\omega + \sum_{\omega' \neq \omega} u(h_{\omega'})\mu^s_{\omega'} = u(q)\mu^s_\omega + \sum_{\omega' \neq \omega} u(h_{\omega'})\mu^s_{\omega'}$$

for all $p, q \in \Delta(X)$ and all $h \in F$. Thus $p[\omega]h \sim^s q[\omega]h$, so that $p[\omega']h \sim^s q[\omega']h$ (for all ω') by Axiom B4. Pick ω' such that $\mu^s_{\omega'} > 0$. Then $u(p)\mu^s_{\omega'} = u(q)\mu^s_{\omega'}$, forcing $u(p) = u(q)$. This holds for all $p, q \in \Delta(X)$, so that $f \sim^s g$ for all $f, g \in F$. This contradicts Axiom B2. □

For an ordered pair of states $E = [\omega, \omega']$ (where $\omega \neq \omega'$), lotteries p, q, and any act h, let $(p, q)Eh$ denote the act f such that $f_\omega = p$, $f_{\omega'} = q$, and $f_{\hat{\omega}} = h_{\hat{\omega}}$ for all $\hat{\omega} \in \Omega \backslash \{\omega, \omega'\}$. Recall that $e \in S$ denotes the signal s where $s_\omega = 1$ for all ω.

Lemma 28. *For every $E = [\omega, \omega']$, there are acts f, g, h such that $fEh \sim^e gEh$, $u(g_\omega) - u(f_\omega) > 0$, and $u(f_{\omega'}) - u(g_{\omega'}) > 0$.*

Proof. By Lemma 27, $\frac{\mu^e_\omega}{\mu^e_{\omega'}} := \delta$ is well-defined. Suppose $\delta \geq 1$. Let p, p' be interior such that $u(p) - u(p') > 0$. Take $f_{\omega'} = p$, $g_{\omega'} = p'$, $g_\omega = p$, and $f_\omega = \alpha p + (1 - \alpha)p'$. Then $u(g_\omega) - u(f_\omega) > 0$ and $u(f_{\omega'}) - u(g_{\omega'}) > 0$ for all $\alpha \in [0, 1)$. Moreover,

$$\frac{u(f_{\omega'}) - u(g_{\omega'})}{u(g_\omega) - u(f_\omega)} = \frac{u(p) - u(p')}{u(p) - \alpha u(p) - (1 - \alpha)u(p')} = \frac{1}{1 - \alpha}$$

Now let $\alpha = \frac{\delta - 1}{\delta}$, so that $\frac{1}{1-\alpha} = \delta = \frac{\mu^e_\omega}{\mu^e_{\omega'}}$. Since $\delta \geq 1$, we have $\alpha \in [0, 1)$. Thus,

$$\frac{u(f_{\omega'}) - u(g_{\omega'})}{u(g_\omega) - u(f_\omega)} = \frac{\mu^e_\omega}{\mu^e_{\omega'}}$$

so that

$$u(f_\omega)\mu^e_\omega + u(f_{\omega'})\mu^e_{\omega'} = u(g_\omega)\mu^e_\omega + u(g_{\omega'})\mu^e_{\omega'}.$$

Therefore $fEh \sim^e gEh$ for all h. The proof for $\delta \leq 1$ is similar. $\square$

Lemma 29. *If $f \sim^s g$, $\alpha \in (0,1)$ and $t = sE(\alpha s)$, then $(\alpha f + (1-\alpha)h)Ef \sim^t (\alpha g + (1-\alpha)h)Eg$ for all h.*

Proof. First, suppose toward a contradiction that

$$(\alpha f + (1-\alpha)h)Ef \succ^t (\alpha g + (1-\alpha)h)Eg \tag{12}$$

for some h. By the expected utility representation for $\succsim^t$, (12) holds for every choice of h. In particular, $h = f$ gives

$$f \succ^t (\alpha g + (1-\alpha)f)Eg \tag{13}$$

Take $F = \Omega\backslash E$ and let $t' = tF(\alpha t)$. Notice that $t' = (\alpha t)Et = (\alpha s)E(\alpha s) = \alpha s$. Applying Axiom B5 to (13) with F and t' gives

$$(\alpha f + (1-\alpha)h')Ff \succ^{\alpha s} \Big[\alpha[(\alpha g + (1-\alpha)f)Eg] + (1-\alpha)h'\Big]F\Big[(\alpha g + (1-\alpha)f)Eg\Big] \ \forall h'$$

Subbing in $h' = f$ yields

$$\begin{aligned} f &\succ^{\alpha s} \Big[(\alpha g + (1-\alpha)f)Eg\Big]E\Big[\alpha[(\alpha g + (1-\alpha)f)Eg] + (1-\alpha)f\Big] \\ &= [\alpha g + (1-\alpha)f]E[\alpha g + (1-\alpha)f] \\ &= \alpha g + (1-\alpha)f \end{aligned}$$

By Continuity (B3), $\succsim^{\alpha s} = \succsim^s$. Thus, $f \succ^s \alpha g + (1-\alpha)f$. Since $\succsim^s$ has an expected utility representation, this contradicts the original assumption that $f \sim^s g$. Thus, $(\alpha g + (1-\alpha)h)Eg \succsim^t (\alpha f + (1-\alpha)h)Ef$ for all h. A similar argument establishes $(\alpha f + (1-\alpha)h)Ef \succsim^t (\alpha g + (1-\alpha)h)Eg$ for all h. $\square$

Lemma 30. *Let $E = [\omega, \omega']$ and $s \in S$ such that $s_\omega > 0$ or $s_{\omega'} > 0$. Then $fEh \sim^e gEh$ implies $(s_{\omega'} g_\omega + (1 - s_{\omega'})f_\omega, g_{\omega'})Eh \sim^s (f_\omega, s_\omega f_{\omega'} + (1-s_\omega)g_{\omega'})Eh$.*

Proof. Let $\alpha = s_\omega$ and $\beta = s_{\omega'}$. Consider $E' = \Omega\backslash\omega$ and $t' = eE'(\alpha s) = (\alpha, 1)[\omega, \omega']e$. Lemma 29 implies

$$[\alpha(fEh) + (1-\alpha)\hat{h}]E'[fEh] \sim^{t'} [\alpha(gEh) + (1-\alpha)\hat{h}]E'[gEh] \ \forall \hat{h}$$

Equivalently, for all $\hat{h}$,

$$(f_\omega, \alpha f_{\omega'} + (1-\alpha)\hat{h}_{\omega'})[\omega, \omega'](\alpha h + (1-\alpha)\hat{h}) \sim^{t'} (g_\omega, \alpha g_{\omega'} + (1-\alpha)\hat{h}_{\omega'})[\omega, \omega'](\alpha h + (1-\alpha)\hat{h})$$

Subbing in $\hat{h} = g$ gives

$$(f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})[\omega, \omega'](\alpha h + (1-\alpha)g) \sim^{t'} (g_\omega, g_{\omega'})[\omega, \omega'](\alpha h + (1-\alpha)g) \tag{14}$$

Using the expected utility representation for $\succsim^{t'}$, (14) clearly holds if, on the complement of E, $\alpha h + (1-\alpha)g$ is replaced with any act. Thus,

$$(f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})Eh \sim^{t'} gEh \tag{15}$$

Now take $E'' = \Omega \backslash \omega'$ and let $t'' = t'E''(\beta t') = (\alpha, \beta)[\omega, \omega']e$. Applying Lemma 29 to (15), E'', and t'' gives $\tilde{g}(\hat{h}) \sim^{t''} \tilde{f}(\hat{h})$ for all $\hat{h}$, where

$$\begin{aligned}\tilde{g}(\hat{h}) &:= (\beta(gEh) + (1-\beta)\hat{h})E''(gEh) \\ &= (gEh)[\omega'](\beta(gEh) + (1-\beta)\hat{h}) \\ &= (\beta g_\omega + (1-\beta)\hat{h}_\omega, g_{\omega'})E(\beta h + (1-\beta)\hat{h})\end{aligned}$$

and

$$\begin{aligned}\tilde{f}(\hat{h}) &:= \Big[\beta[(f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})Eh] + (1-\beta)\hat{h}\Big]E''\Big[(f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})Eh\Big] \\ &= (\alpha f_{\omega'} + (1-\alpha)g_{\omega'})[\omega']\Big[\beta[(f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})Eh] + (1-\beta)\hat{h}\Big] \\ &= (\beta f_\omega + (1-\beta)\hat{h}_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})E(\beta h + (1-\beta)\hat{h})\end{aligned}$$

Thus, substituting $\hat{h} = f$ into $\tilde{g}(\hat{h}) \sim^{t''} \tilde{f}(\hat{h})$ yields

$$(\beta g_\omega + (1-\beta)f_\omega, g_{\omega'})E(\beta h + (1-\beta)f) \sim^{t''} (f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})E(\beta h + (1-\beta)f) \tag{16}$$

Using the expected utility representation, (16) holds if, on E^c, $\beta h + (1-\beta)f$ is replaced with any other act. Thus,

$$(\beta g_\omega + (1-\beta)f_\omega, g_{\omega'})Eh \sim^{t''} (f_\omega, \alpha f_{\omega'} + (1-\alpha)g_{\omega'})Eh \tag{17}$$

Since $\alpha = s_\omega$ and $\beta = s_{\omega'}$, the desired acts are indifferent under signal $t'' = (\alpha, \beta)Ee = sEe$. If $\Omega = \{\omega, \omega'\}$, we are done. Otherwise, there is at least one $\hat{\omega} \neq \omega, \omega'$. We will show that the indifference holds for all signals of the form sEt where $t_{\hat{\omega}} > 0$ for all $\hat{\omega} \neq \omega, \omega'$. Then, a continuity argument will establish indifference under signal s.

Suppose $t_{\hat{\omega}} > 0$. Let $F = \Omega \backslash \hat{\omega}$ and $\hat{t} = t''F(t_{\hat{\omega}}t'') = (\alpha, \beta, t_{\hat{\omega}})[\omega, \omega', \hat{\omega}]e$. Applying Lemma

29 to (17) with F and $\hat{t}$ gives $\hat{f}(\hat{h}) \sim^{\hat{t}} \hat{g}(\hat{h})$ for all $\hat{h}$, where

$$\begin{aligned}\hat{f}(\hat{h}) &:= \Big[t_{\hat{\omega}}[(\beta g_\omega + (1-\beta) f_\omega, g_{\omega'})Eh] + (1-t_{\hat{\omega}})\hat{h}\Big] F \Big[(\beta g_\omega + (1-\beta) f_\omega, g_{\omega'})Eh\Big] \\ &= h_{\hat{\omega}}[\hat{\omega}]\Big[\Big(t_{\hat{\omega}}(\beta g_\omega + (1-\beta) f_\omega) + (1-t_{\hat{\omega}})\hat{h}_\omega, t_{\hat{\omega}} g_{\omega'} + (1-t_{\hat{\omega}})\hat{h}_{\omega'}\Big) E \Big(t_{\hat{\omega}} h + (1-t_{\hat{\omega}})\hat{h}\Big)\Big]\end{aligned}$$

and

$$\begin{aligned}\hat{g}(\hat{h}) &:= \Big[t_{\hat{\omega}}[(f_\omega, \alpha f_{\omega'} + (1-\alpha) g_{\omega'})Eh] + (1-t_{\hat{\omega}})\hat{h}\Big] F \Big[(f_\omega, \alpha f_{\omega'} + (1-\alpha) g_{\omega'})Eh\Big] \\ &= h_{\hat{\omega}}[\hat{\omega}]\Big[\Big(t_{\hat{\omega}} f_\omega + (1-t_{\hat{\omega}})\hat{h}_\omega, t_{\hat{\omega}}(\alpha f_{\omega'} + (1-\alpha) g_{\omega'}) + (1-t_{\hat{\omega}})\hat{h}_{\omega'}\Big) E \Big(t_{\hat{\omega}} h + (1-t_{\hat{\omega}})\hat{h}\Big)\Big]\end{aligned}$$

Using these expressions together with the expected utility representation for $\succsim^{\hat{t}}$ yields

$$t_{\hat{\omega}}\mu^{\hat{t}}_\omega[\beta u(g_\omega) + (1-\beta)u(f_\omega)] + t_{\hat{\omega}}\mu^{\hat{t}}_{\omega'}u(g_{\omega'}) = t_{\hat{\omega}}\mu^{\hat{t}}_\omega u(f_\omega) + t_{\hat{\omega}}\mu^{\hat{t}}_{\omega'}[\alpha u(f_{\omega'}) + (1-\alpha)u(g_{\omega'})]$$

Since $t_{\hat{\omega}} > 0$, we may cancel $t_{\hat{\omega}}$ and add $\sum_{\omega'' \neq \omega, \omega'} \mu^{\hat{t}}_{\omega''} u(h_{\omega''})$ to both sides. Thus

$$(\beta g_\omega + (1-\beta) f_\omega, g_{\omega'})Eh \sim^{\hat{t}} (f_\omega, \alpha f_{\omega'} + (1-\alpha) g_{\omega'})Eh$$

So, the desired indifference holds at signal $\hat{t} = (\alpha, \beta, t_{\hat{\omega}})[\omega, \omega', \hat{\omega}]e$. If there exists some $\omega^* \in \Omega \backslash \{\omega, \omega', \hat{\omega}\}$, apply the above argument again, this time with $F = \Omega \backslash \omega^*$ and $t^* = \hat{t}F(t_{\omega^*}\hat{t}) = (\alpha, \beta, t_{\hat{\omega}}, t_{\omega^*})[\omega, \omega', \hat{\omega}, \omega^*]e$, where $t_{\omega^*} > 0$. Clearly, repeating this procedure yields the desired indifference for all signals of the form sEt where $t_{\omega''} > 0$ for all $\omega'' \neq \omega, \omega'$.

To see that $(s_{\omega'} g_\omega + (1-s_{\omega'}) f_\omega, g_{\omega'})Eh \sim^s (f_\omega, s_\omega f_{\omega'} + (1-s_\omega) g_{\omega'})Eh$, suppose that one of these acts is strictly preferred over the other at s. By Axiom B3, there is a neighborhood of s such that every signal in the neighborhood yields the same strict ranking. But, as is easily verified, every neighborhood of s in the given topology contains a signal of the form sEt where $t_{\omega''} > 0$ for all $\omega'' \neq \omega, \omega'$. As shown above, such signals yield indifference between the two acts. Thus, indifference must hold at s. □

Lemma 31. *If $s_\omega = 0$, then $\mu^s_\omega = 0$.*

Proof. Since $s_\omega = 0$ and $s \in S$, there is a state $\omega' \neq \omega$ such that $s_{\omega'} > 0$. Let $E = [\omega, \omega']$. By Lemma 28, there are acts f, g, h such that $fEh \sim^e gEh$ and $u(g_\omega) - u(f_\omega) > 0$. Lemma 30 implies

$$\mu^s_\omega[s_{\omega'}u(g_\omega) + (1-s_{\omega'})u(f_\omega)] + \mu^s_{\omega'}u(g_{\omega'}) = \mu^s_\omega u(f_\omega) + \mu^s_{\omega'}[s_\omega u(f_{\omega'}) + (1-s_\omega)u(g_{\omega'})]$$

and so

$$\mu^s_\omega s_{\omega'}[u(g_\omega) - u(f_\omega)] = \mu^s_{\omega'} s_\omega[u(f_{\omega'}) - u(g_{\omega'})].$$

Substituting $s_\omega = 0$ gives

$$\mu^s_\omega s_{\omega'}[u(g_\omega) - u(f_\omega)] = 0.$$

Since $s_{\omega'} > 0$ and $u(g_\omega) - u(f_\omega) > 0$, this implies $\mu^s_\omega = 0$. □

Lemma 32. *If $s_\omega > 0$ and $s_{\omega'} > 0$, then $\frac{\mu^s_\omega}{\mu^s_{\omega'}} = \frac{s_\omega \mu^e_\omega}{s_{\omega'} \mu^e_{\omega'}}$.*

Proof. Let $E = [\omega, \omega']$. By Lemma 28, there are acts f, g, h such that $fEh \sim^e gEh$, $u(g_\omega) - u(f_\omega) > 0$, and $u(f_{\omega'}) - u(g_{\omega'}) > 0$. Moreover, the expected utility representation for $\succsim^e$ implies

$$u(f_\omega)\mu^e_\omega + u(f_{\omega'})\mu^e_{\omega'} = u(g_\omega)\mu^e_\omega + u(g_{\omega'})\mu^e_{\omega'}$$

Since μ^e has full support (Lemma 27), this implies

$$\frac{u(f_{\omega'}) - u(g_{\omega'})}{u(g_\omega) - u(f_\omega)} = \frac{\mu^e_\omega}{\mu^e_{\omega'}}$$

As in the proof of Lemma 31, Lemma 30 implies

$$\mu^s_\omega s_{\omega'}[u(g_\omega) - u(f_\omega)] = \mu^s_{\omega'} s_\omega[u(f_{\omega'}) - u(g_{\omega'})].$$

Since $s_\omega > 0$ and $s_{\omega'} > 0$, Lemma 27 implies $\mu^s_\omega > 0$ and $\mu^s_{\omega'} > 0$. Thus

$$\begin{aligned} \frac{\mu^s_\omega}{\mu^s_{\omega'}} &= \frac{s_\omega}{s_{\omega'}} \frac{u(f_{\omega'}) - u(g_{\omega'})}{u(g_\omega) - u(f_\omega)} \\ &= \frac{s_\omega}{s_{\omega'}} \frac{\mu^e_\omega}{\mu^e_{\omega'}}, \end{aligned}$$

as desired. □

Proof of Theorem 2

To prove the theorem, let $s \in S$ and observe that by Lemmas 27 and 31, $\mu^s_\omega = 0$ if and only if $s_\omega = 0$, as prescribed by Bayes' rule. Combined with Lemma 32, this implies that the ratio $\frac{\mu^s_\omega}{\mu^s_{\omega'}}$ is pinned down for every choice of ω' such that $s_{\omega'} > 0$.

Notice that for any $\lambda > 0$, $\lambda\mu^s := (\lambda\mu^s_\omega)_{\omega\in\Omega}$ yields the same ratios. Thus, μ^s is the unique probability distribution on the ray passing through $\left(\frac{s_\omega \mu^e_\omega}{s_{\omega'} \mu^e_{\omega'}}\right)_{\omega\in\Omega}$ for any choice of ω' such that $s_{\omega'} > 0$. In other words, the ratios $\frac{s_\omega \mu^e_\omega}{s_{\omega'} \mu^e_{\omega'}}$ $(s_{\omega'} > 0)$ pin down a point in projective space, which corresponds to a ray through the origin in $\mathbb{R}^\Omega$. This ray intersects the probability simplex at a unique point. Since the probability distribution given by $\mu^s_\omega = \frac{s_\omega \mu^e_\omega}{\sum_{\omega'} s_{\omega'} \mu^e_{\omega'}}$ is a point on this ray, it must coincide with μ^s. Hence, μ^s is the Bayesian posterior for signal s and prior $\mu := \mu^e$. This completes the proof.

C Proofs for Section 5

Lemma 33. *Let $E, F \subsetneq \Omega$, $E \neq F$, and $s, t \in S$. Then:*

(i) If s and t are EF-equivalent, then $\mu^s(E) > \mu^s(F) \Rightarrow \mu^t(E) > \mu^t(F)$.

(ii) If $\mu^s(E) > \mu^s(F)$ and $\mu^t(E) > \mu^t(F)$, then s and t are EF-equivalent.

Proof of (i). Suppose s and t are EF-equivalent. First, we prove that $\mu^s(E) > \mu^s(F) \Rightarrow \mu^t(E) \geq \mu^t(F)$; the proof is by contradiction.

Suppose $\mu^s(E) > \mu^s(F)$ and $\mu^t(F) > \mu^t(E)$. Choose $p, q \in \Delta X$ such that $u(p) > u(q)$ and $v(p) \neq v(q)$. Let $A = \{pEq, pFq\}$. We will show that for all $\varepsilon > 0$, there exist $s' \in N^\varepsilon(s)$, $t' \in N^\varepsilon(t)$ and an experiment σ with $s', t' \in \sigma$ such that $\sigma \not\sim^A \sigma^{s'+t'}$. In particular, take $\varepsilon > 0$ small enough so that $\mu^{s'}(E) > \mu^{s'}(F)$ and $\mu^{t'}(F) > \mu^{t'}(E)$ for all $s' \in N^\varepsilon(s)$, $t' \in N^\varepsilon(t)$.

For $s' \in N^\varepsilon(s)$ and $t' \in N^\varepsilon(t)$, let $\sigma = [r', s', t']$ (where $r' = e - s' - t'$), so that $\sigma^{s'+t'} = [r', s' + t']$. Assume that $r' \neq 0$ (if $r' = 0$, take $\sigma = [s', t']$ instead; the proof for this case is similar). By hypothesis, $V^A(\sigma) = V^A(\sigma^{s'+t'})$. Abusing notation slightly, we may write $V^A(\sigma) = V^A(r') + V^A(s') + V^A(t')$ where, for arbitrary signals $\hat{s}$,

$$V^A(\hat{s}) := \sum_{\omega \in \Omega} \hat{s}_\omega \nu_\omega v(\overline{f}^{\hat{s}}_\omega)$$

where $\overline{f}^{\hat{s}} \in \Delta c^{\hat{s}}(A)$ is the act at $\hat{s}$ that yields sender-preferred tie-breaking. Similarly, we may write $V^A(\sigma^{s'+t'}) = V^A(r') + V^A(s' + t')$. Thus, $\sigma \sim^A \sigma^{s'+t'}$ if and only if $V^A(s') + V^A(t') = V^A(s' + t')$ (the same condition would hold if $r' = 0$). Observe that

$$\begin{aligned} V^A(s') + V^A(t') &= \left[\sum_{\omega \in E} s'_\omega \nu_\omega v(p) + \sum_{\omega' \in E^c} s'_{\omega'} \nu_{\omega'} v(q)\right] + \left[\sum_{\omega \in F} t'_\omega \nu_\omega v(p) + \sum_{\omega' \in F^c} t_{\omega'} \nu_{\omega'} v(q)\right] \\ &= v(p) \left[\sum_{\omega \in E} s'_\omega \nu_\omega + \sum_{\omega \in F} t'_\omega \nu_\omega\right] + v(q) \left[\sum_{\omega' \in E^c} s'_{\omega'} \nu_{\omega'} + \sum_{\omega' \in F^c} t'_{\omega'} \nu_{\omega'}\right] \end{aligned}$$

Now consider $V^A(s' + t')$. There are three cases:

(1) $c^{s'+t'}(A) = pEq$. This means $\mu^{s'+t'}(E) > \mu^{s'+t'}(F)$. Then

$$V^A(s'+t') = \sum_{\omega \in E}(s'_\omega + t'_\omega)\nu_\omega v(p) + \sum_{\omega' \in E^c}(s'_{\omega'} + t'_{\omega'})\nu_{\omega'} v(q)$$

and $\sigma \sim^A \sigma^{s'+t'}$ if and only if

$$v(p)\left[\sum_{\omega \in F} t'_\omega \nu_\omega - \sum_{\omega \in E} t'_\omega \nu_\omega\right] + v(q)\left[\sum_{\omega' \in F^c} t'_{\omega'}\nu_{\omega'} - \sum_{\omega' \in E^c} t'_{\omega'}\nu_{\omega'}\right] = 0$$

$$\Leftrightarrow \quad v(p)\sum_{\omega \in F} t'_\omega \nu_\omega + v(q)\sum_{\omega' \in F^c} t'_{\omega'}\nu_{\omega'} = v(p)\sum_{\omega \in E} t'_\omega \nu_\omega + v(q)\sum_{\omega' \in E^c} t'_{\omega'}\nu_{\omega'}$$

In other words, DM1 must be indifferent between pEq and pFq at signal t'. Since $v(p) \neq v(q)$, this is equivalent to $\nu^{t'}(E) = \nu^{t'}(F)$; that is, $\sum_{\omega \in E} t'_\omega \nu_\omega - \sum_{\omega \in F} t'_\omega \nu_\omega = 0$. There is a hyperplane (in S) of such t'. Clearly, then, $N^\varepsilon(t)$ contains a signal t' that is not on that hyperplane. Hence, for small enough ε, $N^\varepsilon(t)$ contains a t' such that $\sigma \not\sim^A \sigma^{s'+t'}$.

(2) $c^{s'+t'}(A) = pFq$. This means $\mu^{s'+t'}(F) > \mu^{s'+t'}(E)$. Similar algebra to case (1) shows that $\sigma \sim^A \sigma^{s'+t'}$ if and only if $\nu^{s'}(E) = \nu^{s'}(F)$. Hence, for all $\varepsilon > 0$ sufficiently small, there exists $s' \in N^\varepsilon(s)$ such that $\sigma \not\sim^A \sigma^{s'+t'}$.

(3) $c^{s'+t'}(A) = \{pEq, pFq\}$. This means $\mu^{s'+t'}(E) = \mu^{s'+t'}(F)$. Clearly, small perturbations of s' and t' yield $\mu^{s'+t'}(E) \neq \mu^{s'+t'}(F)$, bringing us to either case (1) or case (2).

Thus, in all cases, we have $s' \in N^\varepsilon(s)$ and $t' \in N^\varepsilon(t)$ such that $\sigma \not\sim^A \sigma^{s'+t'}$ for ε sufficiently small, contradicting EF-equivalence.

We have shown that if s and t are EF-equivalent and $\mu^s(E) > \mu^s(F)$, then $\mu^t(E) \geq \mu^t(F)$. To establish that $\mu^t(E) > \mu^t(F)$, suppose toward a contradiction that $\mu^t(E) = \mu^t(F)$. Then every neighborhood $N^\varepsilon(t)$ contains a signal t' such that $\mu^{t'}(F) > \mu^{t'}(E)$. As shown above, this implies that $N^\varepsilon(t)$ contains a t' such that $\sigma \not\sim^A \sigma^{s+t'}$ for some EF-bet A and experiment σ with $s, t' \in \sigma$. This contradicts EF-equivalence of s and t. Thus, $\mu^t(E) > \mu^t(F)$. □

Proof of (ii). Suppose $\mu^s(E) > \mu^s(F)$ and $\mu^t(E) > \mu^t(F)$, and let $A = \{pEq, pFq\}$ be an EF-menu. Then $c^s(A) = c^t(A) = \{f\}$ for some $f \in A$. It follows immediately that $\sigma \sim^A \sigma^{s+t}$ for all σ such that $s, t \in \sigma$. Thus, s and t are EF-equivalent.

□

Proof of Proposition 1

Proof that (i) implies (ii). Suppose $\mu = \dot{\mu}$. Let s and t be EF-equivalent for some E and F. By part (i) of Lemma 33, μ^s and μ^t give the same strict ranking of E and F. Since $\mu = \dot{\mu}$, the same strict ranking is given by $\dot{\mu}^s$ and $\dot{\mu}^t$. Thus, by part (ii) of Lemma 33, s and t are $\dot{EF}$-equivalent. □

Clearly, (ii) implies (iii).

Proof that (iii) implies (i). First, let $E, F \subseteq \Omega$ satisfy $E \not\subseteq F$ and $F \not\subseteq E$. Then there are signals $s^E, s^F \in S$ such that $\mu^{s^E}(E) > \mu^{s^E}(F)$ and $\mu^{s^F}(F) > \mu^{s^F}(E)$. For any set $Z \subseteq S$, let cl$Z$ denote the closure of Z. By Lemma 33, we have

$$\mathrm{cl}\{t \in S : s^E \text{ is } EF\text{-equivalent to } t\} = \{t \in S : \mu^t(E) \geq \mu^t(F)\}$$

and

$$\mathrm{cl}\{t \in S : s^F \text{ is } EF\text{-equivalent to } t\} = \{t \in S : \mu^t(F) \geq \mu^t(E)\}$$

Since EF-equivalence implies $\dot{EF}$-equivalence, it follows that

$$\{t \in S : \mu^t(E) \geq \mu^t(F)\} = \{t \in S : \dot{\mu}^t(E) \geq \dot{\mu}^t(F)\}$$

and

$$\{t \in S : \mu^t(F) \geq \mu^t(E)\} = \{t \in S : \dot{\mu}^t(F) \geq \dot{\mu}^t(E)\}$$

In particular,

$$H := \{t \in S : \mu^t(E) = \mu^t(F)\} = \{t \in S : \dot{\mu}^t(E) = \dot{\mu}^t(F)\}$$

Every $t \in H$ satisfies the equation $\sum_{\omega \in E} t_\omega \mu_\omega - \sum_{\omega \in F} t_\omega \mu_\omega = 0$. Interpret this as a linear equation in $\mu = (\mu_\omega)_{\omega \in \Omega} \in \mathbb{R}^\Omega$ with coefficients given by t. Thus, every $t \in H$ determines a hyperplane H^t of solutions μ. Since H is a hyperplane in S, we can find $|\Omega| - 1$ linearly independent vectors $\{t^1, \ldots, t^{|\Omega|-1}\} \subseteq H$. By linear independence, the corresponding hyplerplanes H^{t^i} $(i = 1, \ldots, |\Omega| - 1)$ intersect to form a line passing through the origin. This line must intersect $\Delta\Omega$ at a unique point. Thus, $\mu = \dot{\mu}$. □

Proof of Proposition 2

Proof that (i) implies (ii). Let $A = \{pEq, pFq\}$ and suppose $\sigma = [s, t]$ and $\sigma' = [s', t']$ are equivalent experiments that neither $\succsim^A$ nor $\dot{\succsim}^A$ rank indifferent to e. Furthermore, suppose $\sigma \sim^A \sigma'$; we want to show that $\sigma \dot{\sim}^A \sigma'$.

Note that both DM1 and $\dot{\text{DM}}1$ must not be indifferent between p and q (that is, we must have $v(p) \neq v(q)$ and $\dot{v}(p) \neq \dot{v}(q)$). Otherwise, both $\succsim^A$ and $\dot{\succsim}^A$ would rank σ and σ' indifferent to e, regardless of the priors and preferences of DM2 and $\dot{\text{DM}}2$. We also must have (without loss of generality) that $\mu^s(E) > \mu^s(F)$ and $\mu^t(E) < \mu^t(F)$; otherwise, $\succsim^A$ ranks σ indifferent to e.[26] Similarly, $\mu^{s'}$ and $\mu^{t'}$ yield opposite strict rankings of events E and F; since σ and σ' are equivalent, the rankings are $\dot{\mu}^s(E) > \dot{\mu}^s(F)$ and $\dot{\mu}^t(E) < \dot{\mu}^t(F)$.

It will be convenient to write σ' in the form $\sigma' = [s + \delta, t - \delta]$, where $\delta \in \mathbb{R}^\Omega$. Clearly such a δ exists.

Since $\sigma \sim^A \sigma'$, we have $V^A(\sigma) = V^A(\sigma')$. We will use this equality to derive an equation that depends only on ν and δ. Writing down formulas for $V^A(\sigma)$ and $V^A(\sigma')$, in principle, requires several cases depending on how u and v rank lotteries p and q. For our purposes, it will be enough to assume that DM1 behaves as if pEq is chosen at s and s' while pFq is chosen at t and t'. This is so because if $u(p) > u(q)$, then the specified acts are chosen at these signals. If instead $u(p) > u(q)$, then the opposite acts are chosen, but the derivation of our equation is not affected by swapping the roles of p and q. If $u(p) = u(q)$, then DM1 (through sender-preferred tie-breaking) gets his most-preferred act at any signal. The assumption that σ and σ' are not ranked indifferent to e means that, in this case, different acts are chosen at s and t under this tie-breaking criterion. Swapping these acts to correspond with DM1's (strict) ranking of p and q will not affect the algebraic derivation below.

So, suppose pEq is chosen at s and s' while pFq is chosen at t and t'. Given this choice rule, the induced acts, from the perspective of DM1, are

$$f^A_\omega(\sigma) = \begin{cases} s_\omega p + t_\omega q & \text{if } \omega \in E \backslash F \\ s_\omega q + t_\omega p & \text{if } \omega \in F \backslash E \\ p & \text{if } \omega \in E \cap F \\ q & \text{if } \omega \in E^c \cap F^c \end{cases}$$

and

$$f^A_\omega(\sigma') = \begin{cases} (s_\omega + \delta_\omega)p + (t_\omega - \delta_\omega)q & \text{if } \omega \in E \backslash F \\ (s_\omega + \delta_\omega)q + (t_\omega - \delta_\omega)p & \text{if } \omega \in F \backslash E \\ p & \text{if } \omega \in E \cap F \\ q & \text{if } \omega \in E^c \cap F^c \end{cases}$$

[26] If μ^s and μ^t yield the same strict ranking of E and F, then $c^s(A) = c^t(A)$ is a singleton and $\sigma \sim^A e$. If at least one of the rankings is an equality, then $\sigma \sim^A e$ by sender-preferred tie-breaking.

Thus, we may write $V^A(\sigma) = Zv(p) + (1 - Z)v(q)$, where

$$Z := \sum_{\omega \in E \setminus F} \nu_\omega s_\omega + \sum_{\omega \in F \setminus E} \nu_\omega t_\omega + \sum_{\omega \in E \cap F} \nu_\omega$$

This follows from the fact that $V^A(\sigma)$ is the subjective expected utility of $f^A(\sigma)$ according to prior ν and utility index v. Similarly, $V^A(\sigma') = Z'v(p) + (1 - Z')v(q)$, where

$$Z' := \sum_{\omega \in E \setminus F} \nu_\omega (s_\omega + \delta_\omega) + \sum_{\omega \in F \setminus E} \nu_\omega (t_\omega - \delta_\omega) + \sum_{\omega \in E \cap F} \nu_\omega$$

Notice that $Z' = Z + Z^\delta$, where

$$Z^\delta := \sum_{\omega \in E \setminus F} \nu_\omega \delta_\omega - \sum_{\omega \in F \setminus E} \nu_\omega \delta_\omega$$

Now, straightforward algebra establishes that $V^A(\sigma) = V^A(\sigma') \Leftrightarrow [v(p) - v(q)]Z^\delta = 0$. Thus, $\sigma \sim^A \sigma'$ if and only if

$$\sum_{\omega \in E \setminus F} \nu_\omega \delta_\omega - \sum_{\omega \in F \setminus E} \nu_\omega \delta_\omega = 0 \tag{18}$$

This expression would hold even if the roles of p and q were reversed. Since $\nu = \dot{\nu}$, we may substitute $\dot{\nu}$ in place of ν in expression (18) and perform the reverse derivation (observe that all steps are equivalences) to get that $\dot{V}^A(\sigma) = \dot{V}^A(\sigma')$. Thus, $\sigma \mathrel{\dot{\sim}}^A \sigma'$. □

Proof that (ii) implies (i). We prove the contrapositive statement. So, suppose $\nu \neq \dot{\nu}$. Let $E \subsetneq \Omega$ be nonempty and $A = \{pEq, pE^cq\}$ be a bet (take $F = E^c$) such that none of the functions $u, \dot{u}, v, \dot{v}$ rank p indifferent to q.[27] We will find equivalent binary experiments σ, σ' that neither $\succsim^A$ nor $\dot{\succsim}^A$ rank indifferent to e. These experiments will be ranked indifferent by $\succsim^A$ but not by $\dot{\succsim}^A$, thus establishing a violation of (ii).

Let s and t be interior signals (that is, $0 < s_\omega, t_\omega < 1$ for all ω) such that $s + t = e$, $\mu^s(E) > \mu^s(E^c)$, $\dot{\mu}^s(E) > \dot{\mu}^s(E^c)$, $\mu^t(E) < \mu^t(E^c)$, and $\dot{\mu}^t(E) < \dot{\mu}^t(E^c)$. This, too, is easy to achieve. First, let $\hat{s}$ be an indicator signal for E: $s_\omega = 1$ for all $\omega \in E$, $s_\omega = 0$ for all $\omega \in E^c$. Similarly, let $\hat{t} = e - \hat{s}$ be the indicator signal for E^c. Clearly, $\hat{s}$ and $\hat{t}$ satisfy the inequalities above. Since the inequalities are strict, we may perturb $\hat{s}$ and $\hat{t}$ to obtain interior signals $s = \hat{s} + \delta$, $t = \hat{t} - \delta$ for some $\delta \in \mathbb{R}^\Omega$; for $\|\delta\|$ sufficiently small (and with appropriate entries negative), s and t are well-defined signals and the above inequalities are

[27] This is easy to achieve: pick an interior lottery p, and then any q that does not lie on any indifference curve through p according to any of the four functions. The four indifference curves occupy a set of measure zero in ΔX, so there are plenty of such q.

satisfied. Since $s+t=e$, $\sigma=[s,t]$ is a well-defined binary experiment.

Notice that $\sigma \sim^A e$ if and only if either

$$\sum_{\omega\in E}\nu_\omega s_\omega - \sum_{\omega\in E^c}\nu_\omega s_\omega = 0 \qquad \text{or} \qquad \sum_{\omega\in E}\nu_\omega t_\omega - \sum_{\omega\in E^c}\nu_\omega t_\omega = 0$$

This follows from straightforward algebra (the two cases correspond to whether pEq or qEp is chosen by DM2 at signal e). Observe that if $\sigma \sim^A e$, then arbitrarily small perturbations of s and t will break the indifference. Similar statements hold with $\dot\nu$ in place of ν. So, replacing s and t with such perturbations if necessary, we may assume that neither $\succsim^A$ nor $\dot\succsim^A$ rank σ indifferent to e.

Now let σ' be an experiment of the form $\sigma'=[s+\delta, t-\delta]$ for some $\delta\in\mathbb{R}^\Omega$. When $\|\delta\|$ is sufficiently small, $s+\delta$ and $t-\delta$ are well-defined signals such that $c^s(A)=c^{s+\delta}(A)\neq c^t(A)=c^{t-\delta}(A)$. We need to choose δ such that (a) $\succsim^A$ does not rank σ' indifferent to e; (b) $\sigma\sim^A\sigma'$; (c) $\dot\succsim^A$ does not rank σ' indifferent to e; and (d) $\dot\succsim^A$ does not rank σ indifferent to σ'.

By the derivation in the first part of the proof, $\sigma\sim^A\sigma'$ if and only if

$$\sum_{\omega\in E}\nu_\omega\delta_\omega - \sum_{\omega\in E^c}\nu_\omega\delta_\omega = 0 \tag{19}$$

Given ν, this defines a hyperplane of points δ that satisfy (b) (subject, of course, to $\|\delta\|$ sufficiently small).

Observe that if $\sigma\sim^A\sigma'$, then (since $\succsim^A$ does not rank σ indifferent to e), we have

$$\begin{aligned}\sum_{\omega\in E}\nu_\omega(s_\omega+\delta_\omega) - \sum_{\omega\in E^c}\nu_\omega(s_\omega+\delta_\omega) &= \sum_{\omega\in E}\nu_\omega s_\omega - \sum_{\omega\in E^c}\nu_\omega s_\omega + \sum_{\omega\in E}\nu_\omega\delta_\omega - \sum_{\omega\in E^c}\nu_\omega\delta_\omega\\ &= \sum_{\omega\in E}\nu_\omega s_\omega - \sum_{\omega\in E^c}\nu_\omega s_\omega\\ &\neq 0\end{aligned}$$

Similarly, one can derive $\sum_{\omega\in E}\nu_\omega(t_\omega-\delta_\omega)-\sum_{\omega\in E^c}\nu_\omega(t_\omega-\delta_\omega)\neq 0$. Thus, any δ satisfying (19) (with $\|\delta\|$ sufficiently small) yields a σ' satisfying conditions (a) and (b).

For condition (c), observe that $\sigma'\,\dot\sim^A\,e$ if and only if

$$\sum_{\omega\in E}\dot\nu_\omega(s_\omega+\delta_\omega) - \sum_{\omega\in E^c}\dot\nu_\omega(s_\omega+\delta_\omega) = 0 \qquad \text{or} \qquad \sum_{\omega\in E}\dot\nu_\omega(t_\omega-\delta_\omega) - \sum_{\omega\in E^c}\dot\nu_\omega(t_\omega-\delta_\omega) = 0$$

Again, this follows from straightforward algebra, and the two cases correspond to whether

pEq or pE^cq is chosen by $\dot{\text{DM}}2$ at signal e. The left-hand equation can be rewritten as:

$$\left[\sum_{\omega\in E}\dot{\nu}_\omega s_\omega - \sum_{\omega\in E^c}\dot{\nu}_\omega s_\omega\right] + \left[\sum_{\omega\in E}\dot{\nu}_\omega \delta_\omega - \sum_{\omega\in E^c}\dot{\nu}_\omega \delta_\omega\right] = 0$$

The first term is nonzero because $\dot{\succsim}^A$ does not rank σ indifferent to e. Hence, the set of all δ satisfying this equality is bounded away from $\delta = 0$ (the solutions form a plane that does not pass through the origin). Hence, for $\|\delta\|$ small, the equality is not satisfied. A similar expression holds for the right-hand equation above (involving t). Thus, for $\|\delta\|$ sufficiently small, condition (c) is satisfied.

For condition (d), observe that $\sigma \mathrel{\dot{\sim}^A} \sigma'$ requires

$$\sum_{\omega\in E}\dot{\nu}_\omega \delta_\omega - \sum_{\omega\in E^c}\dot{\nu}_\omega \delta_\omega = 0$$

Comparing this to (19), we see that δ must belong to the kernel of two linear functions; specifically,

$$L : \delta \mapsto \sum_{\omega\in E}\nu_\omega \delta_\omega - \sum_{\omega\in E^c}\nu_\omega \delta_\omega \qquad \text{and} \qquad \dot{L} : \delta \mapsto \sum_{\omega\in E}\dot{\nu}_\omega \delta_\omega - \sum_{\omega\in E^c}\dot{\nu}_\omega \delta_\omega$$

However, since $\nu \neq \dot{\nu}$ and $\nu, \dot{\nu} \in \Delta\Omega$, no nonzero δ satisfies $L(\delta) = 0 = \dot{L}(\delta)$. Intuitively, $L = 0$ and $\dot{L} = 0$ define hyperplanes with normals ν and $\dot{\nu}$, respectively. Since $\nu \neq \dot{\nu}$ but $\nu, \dot{\nu} \in \Delta\Omega$, the normals lie on distinct rays through the origin and, therefore, the associated hyperplanes only intersect at $\delta = 0$. Hence, any nonzero δ with $\|\delta\|$ sufficiently small satisfying (19) yields a σ' such that (a)–(d) are satisfied. This completes the proof. □

Proof of Theorem 4

Proof. We will prove that (ii) implies (i) (the converse is obvious).

By Propositions 1 and 2, we have $\mu = \dot{\mu}$ and $\nu = \dot{\nu}$. Observe that if A is a (p, q)-bet and $v(p) = v(q)$, then (by sender-preferred tie-breaking) $\succsim^A$ is degenerate. Conversely, $v(p) = v(q)$ if $\succsim^A$ is degenerate for all (p, q)-bets A (again, by sender-preferred tie-breaking). Thus, the indifference curves of v can be deduced from $(\succsim^A)_{A\in\mathcal{A}}$; in particular, for all lotteries p, $\{q \in \Delta X : v(p) = v(q)\} = \{q \in \Delta X : \forall\ (p, q)\text{-bets } A,\ \succsim^A \text{ is degenerate}\}$. Since $(\succsim^A)_{A\in\mathcal{A}} = (\dot{\succsim}^A)_{A\in\mathcal{A}}$, it follows that v and $\dot{v}$ have the same indifference curves in ΔX.

Now observe that for any (p, q)-bet A, there exists σ such that $\sigma^* \succ^A \sigma$ if and only if v and u agree on the ranking of p and q (without loss of generality, $v(p) > v(q)$ and $u(p) \geq u(q)$). Similarly, there exists σ such that $\sigma \succ^A \sigma^*$ if and only if v and u disagree on the ranking of p and q (without loss of generality, $v(p) > v(q)$ and $u(q) > u(p)$).

Thus, for any p, the sets $\{q \in \Delta X : v \text{ and } u \text{ agree on the ranking of } p \text{ and } q\}$ and $\{q \in \Delta X : v \text{ and } u \text{ disagree on the ranking of } p \text{ and } q\}$ are revealed by $(\succsim^A)_{A\in\mathcal{A}}$. By linearity, the indifference plane $\{q' \in \Delta X : u(p) = u(q')\}$ can be deduced from either set. In particular, the agreement region is an intersection of two half-spaces in ΔX. One half-space is bounded by the indifference curve (plane) for v through p, and the other is bounded by the indifference plane for u through p. As argued above, these planes can be distinguished using DM1's preferences for information (the indifference curve for v through p is revealed by (p,q)-bets A where $\succsim^A$ is degenerate). Hence, the indifference curves of u are revealed by $(\succsim^A)_{A\in\mathcal{A}}$. Since $(\succsim^A)_{A\in\mathcal{A}} = (\dot{\succsim}^A)_{A\in\mathcal{A}}$, it follows that u and $\dot{u}$ have the same indifference curves.

Since the indifference curves of u are pinned down, there are (up to positive affine transformation) two choices for u; call them u and $-u$. These are associated with two choices for v (namely, v and $-v$) because the agreement and disagreement regions are pinned down. Since $(\succsim^A)_{A\in\mathcal{A}} = (\dot{\succsim}^A)_{A\in\mathcal{A}}$, the same possibilities hold for $\dot{\text{D}}$M; that is, either $(\dot{v}, \dot{u}) \approx (v, u)$ or $(\dot{v}, \dot{u}) \approx (-v, -u)$.

We now show that, given the priors μ and ν, only one of the pairs (v, u) or $(-v, -u)$ can be consistent with $(\succsim^A)_{A\in\mathcal{A}}$. Pick $p^1, p^2, p^3 \in \Delta X$ such that $u(p^3) > u(p^2) > u(p^1)$ and $u(p^2) - u(p^1) > u(p^3) - u(p^2)$. We also require $v(p^3) \neq v(p^1)$; this is easily achieved by perturbing p^1 and p^3 along their respective indifference planes for u. Pick any $E = [\omega, \omega']$, $q \in \Delta X$, and $h' \in F$, and let $A = \{f, g, h\}$ where

$$f = (\mu_{\omega'} p^1 + (1 - \mu_{\omega'})q, \mu_\omega p^3 + (1 - \mu_\omega)q)Eh'$$
$$g = (\mu_{\omega'} p^2 + (1 - \mu_{\omega'})q, \mu_\omega p^2 + (1 - \mu_\omega)q)Eh'$$
$$h = (\mu_{\omega'} p^3 + (1 - \mu_{\omega'})q, \mu_\omega p^1 + (1 - \mu_\omega)q)Eh'$$

It is straightforward to verify that DM2's choices from A (under μ and u) induces a symmetric division of S. In particular, $c^s(A) = A$ if $s_\omega = 0 = s_{\omega'}$; $c^s(A) = f$ if $\frac{s_{\omega'}}{s_\omega} > \frac{u(p^2)-u(p^1)}{u(p^3)-u(p^2)}$; $c^s(A) = h$ if $\frac{s_{\omega'}}{s_\omega} < \frac{u(p^3)-u(p^2)}{u(p^2)-u(p^1)}$; and $c^s(A) = g$ if $\frac{u(p^3)-u(p^2)}{u(p^2)-u(p^1)} < \frac{s_{\omega'}}{s_\omega} < \frac{u(p^2)-u(p^1)}{u(p^3)-u(p^2)}$ (these inequalities assume $s_\omega > 0$; at signal $e_{\omega'}$, DM2 chooses f). Since $u(p^2) - u(p^1) > u(p^3) - u(p^2)$, there is an interval of values for $\frac{s_{\omega'}}{s_\omega}$ where one act in A is strictly optimal. In particular, $c^e(A) = g$.

Consider an experiment $\sigma = [s, t]$ where $c^s(A) = c^t(A) = g$, $s_{\hat{\omega}} = t_{\hat{\omega}} = 1/2$ for all $\hat{\omega} \notin E$, and $s \neq t$. This can be done by taking $s = \frac{1}{2}e + \delta$, $t = \frac{1}{2}e - \delta$ for some $\delta \in \mathbb{R}^\Omega$ where $\delta_{\hat{\omega}} = 0$ for all $\hat{\omega} \notin E$. As long as $\|\delta\|$ is sufficiently small, we maintain $c^s(A) = c^t(A) = g$. Then $F^A(\sigma) = g$. Applying another such perturbation yields an experiment $\sigma' = [s', t']$ where $s' = s + \delta'$, $t' = t - \delta'$; for small $\|\delta'\|$, we once again have $F^A(\sigma') = g$, so that $\sigma \sim^A \sigma'$.

Now consider behavior under $(-v, -u)$. Under utility index $-u$, DM2 satisfies $f \succ^s g$

if and only if $s_\omega \mu_\omega(-\mu_{\omega'} u(p^1)) + s_{\omega'}\mu_{\omega'}(-\mu_\omega u(p^3)) > s_\omega \mu_\omega(-\mu_{\omega'} u(p^2)) + s_{\omega'}\mu_{\omega'}(-\mu_\omega u(p^2))$; equivalently, $s_\omega(u(p^2)-u(p^1)) > s_{\omega'}(u(p^3)-u(p^2))$. Similarly, $g \succsim^s h$ if and only if $s_\omega(u(p^3)-u(p^2)) \geq s_{\omega'}(u(p^2)-u(p^1))$. Since $u(p^2)-u(p^1) > u(p^3)-u(p^2)$, we get that $g \succsim^s h \Rightarrow f \succ^s g$. Thus, for all $s \in S$, $g \notin c^s(A)$. Similar algebra establishes that $f \succsim^s h$ iff $s_\omega \geq s_{\omega'}$ and $h \succsim^s f$ iff $s_{\omega'} \geq s_\omega$. Thus, under $-u$, only acts f and h are chosen by DM2 whenever $s_{\hat\omega} = 0$ for all $\hat\omega \notin E$ (if s has support E^c, then $c^s(A) = A$). With $\sigma = [s,t]$ and $\sigma' = [s',t'] = [s+\delta', t-\delta']$ as defined above, we may therefore assume that $c^s(A) = c^{s'}(A) = h$ and $c^t(A) = c^{t'}(A) = f$. The induced act for σ' is given by

$$F^A_{\hat\omega}(\sigma') = \begin{cases} \mu_{\omega'}[(s_\omega + \delta'_\omega)p^1 + (t_\omega - \delta'_\omega)p^3] + (1-\mu_{\omega'})q & \text{if } \hat\omega = \omega \\ \mu_\omega[(s_{\omega'} + \delta'_{\omega'})p^3 + (t_{\omega'} - \delta'_{\omega'})p^1] + (1-\mu_\omega)q & \text{if } \hat\omega = \omega' \\ h'_{\hat\omega} & \text{if } \hat\omega \notin E \end{cases}$$

For $F^A(\sigma)$, set $\delta' = 0$. Straightforward algebra establishes that under $-v$, we have $\sigma \sim^A \sigma'$ if and only if

$$\nu_{\omega'}\mu_\omega\delta'_{\omega'}[v(p^1) - v(p^3)] = \nu_\omega\mu_{\omega'}\delta'_\omega[v(p^1) - v(p^3)]$$

Since $v(p^1) \neq v(p^3)$, this reduces to

$$\nu_{\omega'}\mu_\omega\delta'_{\omega'} = \nu_\omega\mu_{\omega'}\delta'_\omega$$

We are free to perturb δ'_ω and $\delta'_{\omega'}$ as needed because the only constraint on δ' is that $\|\delta'\|$ is small. Thus, there exists σ' such that $\sigma \not\sim^A \sigma'$ under $(-v,-u)$.

Since we have shown that $\sigma \sim^A \sigma'$ under (v,u) for all $\|\delta'\|$ small but $\sigma \not\sim^A \sigma'$ for some such δ' under $(-v,-u)$, we conclude that only one pair—(v,u) or $(-v,-u)$—can be consistent with $(\succsim^A)_{A\in\mathcal{A}}$. Since $(\succsim^A)_{A\in\mathcal{A}} = (\dot\succsim^A)_{A\in\mathcal{A}}$, the proof is complete. □

D Proofs for Section 6

D.1 Proofs for Section 6.1

Proof of Proposition 3

Proof that (i) implies (ii). Suppose $u \approx v$. Without loss of generality, assume that $u = v$.

First, suppose A is a menu such that $\sigma^* \in \mathcal{E}^*(A)$, and let $\sigma \in \mathcal{E}^*(A)$. Then $V^A(\sigma) = \sum_\omega \nu_\omega v(f^\sigma_\omega)$, where $f^\sigma = F^A(\sigma)$. Let $\omega \in \Omega$. Since $\sigma^* \in \mathcal{E}^*(A)$, there is a lottery p_ω such that for all $g \in c^{e_\omega}(A)$, $g_\omega = p_\omega$. Since DM2 has a Bayesian representation, it follows that

$u(p_\omega) \geq u(g_\omega)$ for all $g \in A$. Thus, $v(p_\omega) \geq v(g_\omega)$ for all $g \in A$ as well. It follows that $v(p_\omega) \geq v(f^\sigma_\omega)$ because f^σ_ω is in the convex hull of $\{g_\omega : g \in A\}$. These statements hold for all ω. Thus,

$$V^A(\sigma^*) = \sum_\omega \nu_\omega v(p_\omega) \geq \sum_\omega \nu_\omega v(f^\sigma_\omega) = V^A(\sigma),$$

so that $\sigma^* \succsim^A \sigma$, as desired.

Now suppose A is a menu and σ is an experiment where $\sigma^* \notin \mathcal{E}^*(A)$ or $\sigma \notin \mathcal{E}^*(A)$. Let $\omega \in \Omega$. Observe that if $g, g' \in \Delta c^{e_\omega}(A)$ and $h \in A$, then $u(g_\omega) = u(g'_\omega) \geq u(h_\omega)$ since DM2 has a Bayesian representation. Thus, $\overline{v}_\omega := v(g_\omega) = v(g'_\omega) \geq v(h_\omega)$ as well. Since $\overline{V}^A(\sigma) = \sum_\omega \nu_\omega v(f^\sigma_\omega)$ for some $f^\sigma \in F^A(\sigma)$, and since f^σ_ω is in the convex hull of $\{h_\omega : h \in A\}$, it follows that

$$\overline{V}^A(\sigma^*) = \sum_\omega \nu_\omega \overline{v}_\omega \geq \sum_\omega \nu_\omega v(f^\sigma_\omega) = \overline{V}^A(\sigma),$$

so that $\sigma^* \succsim^A \sigma$, as desired. □

Clearly, (ii) implies (iii). Thus, to complete the proof, we only need to show that (iii) implies (i).

Proof that (iii) implies (i). First, we establish the following claim: for all interior $p, p' \in \Delta X$, $u(p) > u(p')$ implies $v(p) \geq v(p')$.

To prove the claim, suppose $u(p) > u(p')$ and consider the bet $A = \{pEp', p'Ep\}$, where both E and its complement E^c are nonempty. In this menu, σ^* results in lottery p with certainty (at any signal $e_\omega \in \sigma^*$, DM2 selects pEp' if $\omega \in E$, and $p'Ep$ if $\omega \notin E$). Now pick any $\sigma \in \mathcal{E}^*(A)$ where each $s \in \sigma$ belongs to the interior of S (there are infinitely many such σ because $u(p) > u(p')$ and E, E^c are nonempty). Since DM2 chooses between pEp' and $p'Ep$, the induced act $F^A(\sigma)$ has the property that each $F^A_\omega(\sigma)$ is a lottery over p and p' (a second-order lottery). Thus, using probability weights ν, DM1 treats σ as if it were a second-order lottery over p and p'. The assumptions on σ ensure that positive probability is assigned to both p and p'. Then, since $\sigma^* \succsim^A \sigma$, it follows that $v(p) \geq v(p')$. Thus, the claim is proved.

From the claim, it follows that for all interior lotteries p, $\{p' \in \text{int}\Delta X : u(p) > u(p')\} \subseteq \{p' \in \text{int}\Delta X : v(p) \geq v(p')\}$. Since u and v are continuous, we may take closures to obtain $\{p' \in \Delta X : u(p) \geq u(p')\} \subseteq \{p' \in \Delta X : v(p) \geq v(p')\}$. Since u and v are linear, these lower contour sets are half-spaces in ΔX, and p lies on the bounding hyperplane of each half-space. So, the inclusion forces the bounding hyperplanes to have a common normal vector. Thus, $\{p' \in \Delta X : u(p) \geq u(p')\} = \{p' \in \Delta X : v(p) \geq v(p')\}$. This holds for all interior p, and therefore $u \approx v$. $\square$

Proof of Proposition 4

Proof that (i) implies (ii). Suppose $\nu = \mu$ and let $A = \{pEq, qFp\}$. If $u(p) = u(q)$, then $c^s(A) = A$ for all signals s. Then, under the assumption of sender-preferred tie-breaking, $\succsim^A$ satisfies the Blackwell ordering.

If $u(p) \neq u(q)$, suppose without loss of generality that $u(p) > u(q)$. If $v(p) = v(q)$, then $\succsim^A$ is trivially Blackwell monotone. For any signal s, let ν^s and μ^s denote the Bayesian posteriors of ν and μ upon observing s, respectively. Notice that DM2 prefers pEq over pFq at signal s if and only if $\mu^s(E) \geq \mu^s(F)$. There are two cases.

Case 1: $v(p) > v(q)$. Then DM1 prefers pEq over pFq at signal s if and only if $\nu^s(E) \geq \nu^s(F)$. Since $\mu = \nu$, this means DM1's preferences at s coincide with those of DM2. This holds for all s. Therefore, for all σ,

$$V^A(\sigma) = \sum_{\hat{\omega}} \nu_{\hat{\omega}} \sum_{s \in \sigma} s_{\hat{\omega}} v(f^s_{\hat{\omega}}) \text{ s.t. } f^s \in \operatorname*{argmax}_{f \in A} \sum_{\hat{\omega}} \nu^s_{\hat{\omega}} v(f_{\hat{\omega}})$$

In other words, $V^A(\sigma)$ can be written as the expected utility in decision problem A under experiment σ for an expected-utility maximizer with prior ν, utility index v, and Bayesian updating (that is, choices conditional on s are made to maximize expected utility under Bayesian posterior ν^s and utility index v). It follows that $\succsim^A$ satisfies the Blackwell ordering.

Case 2: $v(p) < v(q)$. In this case, DM1's preferences at s are exactly opposite those of DM2. They agree only when $\mu^s(E) = \mu^s(F)$ (and, hence, $\nu^s(E) = \nu^s(F)$), in which case they are both indifferent between pEq and qEp. Therefore, for all σ,

$$-V^A(\sigma) = \sum_{\hat{\omega}} \nu_{\hat{\omega}} \sum_{s \in \sigma} s_{\hat{\omega}} v'(f^s_{\hat{\omega}}) \text{ s.t. } f^s \in \operatorname*{argmax}_{f \in A} \sum_{\hat{\omega}} \nu^s_{\hat{\omega}} v'(f_{\hat{\omega}}) \tag{20}$$

where $v' := -v$. Thus, $\sigma \sqsupseteq \sigma'$ implies $-V^A(\sigma) \geq -V^A(\sigma')$; that is, $\sigma \sqsupseteq \sigma'$ implies $\sigma' \succsim^A \sigma$. $\square$

Proof that (ii) implies (i). Suppose $\mu \neq \nu$ and assume without loss of generality that $\mu(E) > \nu(E)$. Let $A = \{pEq, qEp\}$ where $u(p) > u(q)$. We will prove that $\succsim^A$ is not Blackwell monotone for the case $v(p) > v(q)$; the case $v(p) < v(q)$ is similar. This is sufficient because linearity of u and v ensures that lotteries p, q can be found for which one of these two cases holds.

Since DM1 and DM2 agree on the ranking of p and q, $\succsim^A$ satisfies the Blackwell ordering around σ^*. That is, for small garblings σ of σ^*, we have $\sigma^* \succsim^A \sigma$. Thus, it will suffice to find a pair of experiments σ, σ' where $\sigma \sqsupseteq \sigma'$ but $\sigma' \succ^A \sigma$.

First, note that upon observing signal s, DM2 prefers pEq over qEp if and only if $\sum_{\omega \in E} s_\omega \mu_\omega \geq \sum_{\omega' \in E^c} s_{\omega'} \mu_{\omega'}$, where E^c is the complement of E. A similar statement holds for DM1: if he were allowed to choose from A after observing s, he would prefer pEq over qEp if and only if $\sum_{\omega \in E} s_\omega \nu_\omega \geq \sum_{\omega' \in E^c} s_{\omega'} \nu_{\omega'}$.

We will work with experiments of the form $\sigma = [r \; s \; t]$. Consider the following three properties:

1. $\sum_{\omega \in E} r_\omega \mu_\omega < \sum_{\omega' \in E^c} r_{\omega'} \mu_{\omega'}$ and $\sum_{\omega \in E} r_\omega \nu_\omega < \sum_{\omega' \in E^c} r_{\omega'} \nu_{\omega'}$
2. $\sum_{\omega \in E} s_\omega \mu_\omega > \sum_{\omega' \in E^c} s_{\omega'} \mu_{\omega'}$ and $\sum_{\omega \in E} s_\omega \nu_\omega < \sum_{\omega' \in E^c} s_{\omega'} \nu_{\omega'}$
3. $\sum_{\omega \in E} t_\omega \mu_\omega > \sum_{\omega' \in E^c} t_{\omega'} \mu_{\omega'}$ and $\sum_{\omega \in E} t_\omega \nu_\omega > \sum_{\omega' \in E^c} t_{\omega'} \nu_{\omega'}$

Property 1 says that DM1 and DM2 both strictly prefer qEp at signal r, while property 3 says that they both strictly prefer pEq at signal t. Property 2 says that at signal s, DM2 strictly prefers pEq while DM1 strictly prefers qEp.

It is fairly simple to see that an experiment $\sigma = [r \; s \; t]$ satisfying properties 1–3 exists. Take s to be an interior signal satisfying property 2 (this can be done because $\mu(E) > \nu(E)$). If necessary, replace s with λs for some $\lambda \in (0, 1)$ so that both $r := 1E^c0 - sE^c0$ and $t := 1E0 - sE0$ are well-defined signals. Then $r + s + t = e$ so that σ is a well-defined experiment that satisfies all three properties. In particular, signal r reveals the true state to be in E^c while signal t reveals the state to be in E, as desired.

Let $\hat{\sigma} = [\hat{r} \; \hat{s} \; \hat{t}]$ be an experiment satisfying properties 1–3, with $\hat{r}$ in place of r, $\hat{s}$ in place of s, and $\hat{t}$ in place of t. Let $\gamma \in [0, 1]$ and define $s := \gamma \hat{s}$, $t := \hat{t} + (1 - \gamma)\hat{s}E0$, and $r := \hat{r} + (1 - \gamma)\hat{s}E^c0$. Then

$$\begin{aligned}
r + s + t &= \hat{r}E[\hat{r} + (1-\gamma)\hat{s}] + \gamma\hat{s}E\gamma\hat{s} + [\hat{t} + (1-\gamma)\hat{s}]E\hat{t} \\
&= [\hat{r} + \gamma\hat{s} + \hat{t} + (1-\gamma)\hat{s}]E[\hat{r} + (1-\gamma)\hat{s} + \gamma\hat{s} + \hat{t}] \\
&= \hat{r} + \hat{s} + \hat{t} \\
&= e
\end{aligned}$$

so that r, s, t are well-defined signals and $\sigma = [r \; s \; t]$ is a well-defined experiment.

Next, we show that σ satisfies properties 1–3. For property 1, observe that $\sum_{\omega \in E} r_\omega \mu_\omega = \sum_{\omega \in E} \hat{r}_\omega \mu_\omega$, and that $\sum_{\omega' \in E^c} r_{\omega'} \mu_{\omega'} = \sum_{\omega' \in E^c} [\hat{r}_{\omega'} + (1-\gamma)\hat{s}_{\omega'}]\mu_{\omega'} \geq \sum_{\omega' \in E^c} \hat{r}_{\omega'} \mu_{\omega'}$. Since $\hat{r}$ satisfies property 1, this implies $\sum_{\omega \in E} r_\omega \mu_\omega = \sum_{\omega \in E} \hat{r}_\omega \mu_\omega < \sum_{\omega' \in E^c} \hat{r}_{\omega'} \mu_{\omega'} \leq \sum_{\omega' \in E^c} r_{\omega'} \mu_{\omega'}$, as desired. A similar argument holds with ν in place of μ. Thus, property 1 is satisfied.

Property 2 for s is clearly inherited from $\hat{s}$. For property 3, note that $\sum_{\omega \in E} t_\omega \mu_\omega = \sum_{\omega \in E} [\hat{t}_\omega + (1-\gamma)\hat{s}_\omega]\mu_\omega \geq \sum_{\omega \in E} \hat{t}_\omega \mu_\omega$ and that $\sum_{\omega' \in E^c} t_{\omega'} \mu_{\omega'} = \sum_{\omega' \in E^c} \hat{t}_{\omega'} \mu_{\omega'}$. Then use the fact that $\hat{t}$ satisfies property 3 to get the result. Once again, the same argument holds with

ν in place of μ to get the second statement of property 3. Thus, property 3 is satisfied.

We now construct a garbling σ' of σ such that $\sigma' \succ^A \sigma$. Let $\sigma' = \sigma M$, where M is the garbling matrix given by

$$M = \begin{bmatrix} \beta & 1-\beta & 0 \\ \alpha & 0 & 1-\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

where $\alpha, \beta \in (0,1)$. Thus, $\sigma' = [r' \; s' \; t']$, where $r' = (1-\beta)r$, $s' = \alpha s + \beta r$, and $t' = (1-\alpha)s + t$. We will choose γ, α, and β such that σ' satisfies properties 1,3, and 2', where:

2'. $\sum_{\omega \in E} s'_\omega \mu_\omega < \sum_{\omega' \in E^c} s'_{\omega'} \mu_{\omega'}$ and $\sum_{\omega \in E} s'_\omega \nu_\omega < \sum_{\omega' \in E^c} s'_{\omega'} \nu_{\omega'}$

Thus, like r and t, signals r' and t' involve DM2 choosing qEp and pEq, respectively, and DM1 agrees with these decisions. But s' now involves DM2 choosing qEp (instead of pEq, which signal s induces), and DM1 agrees with this choice. Thus, M transforms s from a point of disagreement into a point of agreement, s'. This will provide a boost to DM1's ex-ante expected utility large enough to yield $\sigma' \succ^A \sigma$.

Since $r' = (1-\beta)r$ and r satisfies property 1, so does r'. For property 3, the fact that $\sum_{\omega \in E} s_\omega \mu_\omega > \sum_{\omega' \in E^c} s_{\omega'} \mu_{\omega'}$ and $\sum_{\omega \in E} t_\omega \mu_\omega > \sum_{\omega' \in E^c} t_{\omega'} \mu_{\omega'}$, together with $t' = (1-\alpha)s + t$, immediately implies that $\sum_{\omega \in E} t'_\omega \mu_\omega > \sum_{\omega' \in E^c} t'_{\omega'} \mu_{\omega'}$. For the other claim of property 3, we need to show that $\sum_{\omega \in E} t'_\omega \nu_\omega > \sum_{\omega' \in E^c} t'_{\omega'} \nu_{\omega'}$. The left-hand side can be rewritten as $\sum_{\omega \in E} [(1-\alpha)\gamma \hat{s}_\omega + \hat{t}_\omega + (1-\gamma)\hat{s}_\omega] \nu_\omega$; this is a continuous function of γ that converges to $\sum_{\omega \in E} (\hat{t}_\omega + \hat{s}_\omega) \nu_\omega$ as $\gamma \to 0$. The right-hand side can be rewritten as $\sum_{\omega' \in E^c} [(1-\alpha)\gamma \hat{s}_{\omega'} + \hat{t}_{\omega'}]$; this is a continuous function of γ that converges to $\sum_{\omega' \in E^c} \hat{t}_{\omega'} \nu_{\omega'}$ as $\gamma \to 0$. Since $\hat{t}$ satisfies property 3, we get that $\sum_{\omega \in E} [\hat{t}_\omega + \hat{s}_\omega] \nu_\omega \geq \sum_{\omega \in E} \hat{t}_\omega \nu_\omega > \sum_{\omega' \in E^c} \hat{t}_{\omega'} \nu_{\omega'}$. Thus, for sufficiently small γ, we have $\sum_{\omega \in E} t'_\omega \nu_\omega > \sum_{\omega' \in E^c} t'_{\omega'} \nu_{\omega'}$, as desired.

For property 2', the fact that $\sum_{\omega \in E} s_\omega \nu_\omega < \sum_{\omega' \in E^c} s_{\omega'} \nu_{\omega'}$ and $\sum_{\omega \in E} r_\omega \nu_\omega < \sum_{\omega' \in E^c} r_{\omega'} \nu_{\omega'}$, together with $s' = \alpha s + \beta r$, immediately implies that $\sum_{\omega \in E} s'_\omega \nu_\omega < \sum_{\omega' \in E^c} s'_{\omega'} \nu_{\omega'}$. For the other claim of property 2', we need to show that $\sum_{\omega \in E} s'_\omega \mu_\omega < \sum_{\omega' \in E^c} s'_{\omega'} \mu_{\omega'}$. The left-hand side can be rewritten as $\sum_{\omega \in E} [\alpha\gamma \hat{s}_\omega + \beta \hat{r}_\omega] \mu_\omega$; this is a continuous function of γ that converges to $\sum_{\omega \in E} \beta \hat{r}_\omega \mu_\omega$ as $\gamma \to 0$. The right-hand side can be rewritten as $\sum_{\omega' \in E^c} [\alpha\gamma \hat{s}_{\omega'} + \beta(\hat{r}_{\omega'} + (1-\gamma)\hat{s}_{\omega'})] \mu_{\omega'}$; this is a continuous function of γ that converges to $\sum_{\omega' \in E^c} \beta(\hat{r}_{\omega'} + \hat{s}_{\omega'}) \mu_{\omega'}$ as $\gamma \to 0$. Since $\hat{r}$ satisfies property 1, we have $\sum_{\omega \in E} \hat{r}_\omega \mu_\omega < \sum_{\omega' \in E^c} \hat{r}_{\omega'} \mu_{\omega'} \leq \sum_{\omega' \in E^c} (\hat{r}_{\omega'} + \hat{s}_{\omega'}) \mu_{\omega'}$, so that $\sum_{\omega \in E} \beta \hat{r}_\omega \mu_\omega < \sum_{\omega' \in E^c} \beta(\hat{r}_{\omega'} + \hat{s}_{\omega'}) \mu_{\omega'}$. Thus, for small enough γ, we have $\sum_{\omega \in E} s'_\omega \mu_\omega < \sum_{\omega' \in E^c} s'_{\omega'} \mu_{\omega'}$, as desired.

The above arguments show that for γ sufficiently close to zero (but still strictly positive), σ satisfied properties 1–3 and $\sigma' := \sigma M$ satisfies properties 1, 2', and 3. This holds for any choice of $\alpha, \beta \in (0,1)$. For the remainder of the proof, fix such a γ.

It will suffice to show that $V^A(\sigma') > V^A(\sigma)$. Recall that $c^r(A) = qEp$, $c^s(A) = pEq$,

$c^t(A) = pEq$, $c^{r'}(A) = qEp$, $c^{s'}(A) = qEp$, and $c^{t'}(A) = pEq$, and that $r' = (1-\beta)r$, $s' = \alpha s + \beta r$, and $t' = (1-\alpha)s + t$. Therefore,

$$\begin{aligned}
V^A(\sigma) &= \sum_{\omega \in E} \nu_\omega [r_\omega v(q) + s_\omega v(p) + t_\omega v(p)] + \sum_{\omega' \in E^c} \nu_{\omega'} [r_{\omega'} v(p) + s_{\omega'} v(q) + t_{\omega'} v(q)] \\
&= \left[\sum_{\omega \in E} \nu_\omega r_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} r_{\omega'} v(p) \right] + \left[\sum_{\omega \in E} \nu_\omega s_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(q) \right] \\
&\quad + \left[\sum_{\omega \in E} \nu_\omega t_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} t_{\omega'} v(q) \right]
\end{aligned}$$

and

$$\begin{aligned}
V^A(\sigma') &= \sum_{\omega \in E} \nu_\omega [r'_\omega v(q) + s'_\omega v(q) + t'_\omega v(p)] + \sum_{\omega' \in E^c} \nu_{\omega'} [r'_{\omega'} v(p) + s'_{\omega'} v(p) + t'_{\omega'} v(q)] \\
&= \left[\sum_{\omega \in E} \nu_\omega r'_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} r'_{\omega'} v(p) \right] + \left[\sum_{\omega \in E} \nu_\omega s'_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} s'_{\omega'} v(p) \right] \\
&\quad + \left[\sum_{\omega \in E} \nu_\omega t'_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} t'_{\omega'} v(q) \right] \\
&= (1-\beta) \left[\sum_{\omega \in E} \nu_\omega r_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} r_{\omega'} v(p) \right] \\
&\quad + \left[\sum_{\omega \in E} \nu_\omega (\alpha s_\omega + \beta r_\omega) v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} (\alpha s_{\omega'} + \beta r_{\omega'}) v(p) \right] \\
&\quad + \left[\sum_{\omega \in E} \nu_\omega ((1-\alpha) s_\omega + t_\omega) v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} ((1-\alpha) s_{\omega'} + t_{\omega'}) v(q) \right] \\
&= \left[\sum_{\omega \in E} \nu_\omega r_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} r_{\omega'} v(p) \right] + \alpha \left[\sum_{\omega \in E} \nu_\omega s_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(p) \right] \\
&\quad + (1-\alpha) \left[\sum_{\omega \in E} \nu_\omega s_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(q) \right] + \left[\sum_{\omega \in E} \nu_\omega t_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} t_{\omega'} v(q) \right]
\end{aligned}$$

Thus,

$$V^A(\sigma') - V^A(\sigma) = \alpha \left[\sum_{\omega \in E} \nu_\omega s_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(p) \right] - \alpha \left[\sum_{\omega \in E} \nu_\omega s_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(q) \right]$$

It follows that $V^A(\sigma') > V^A(\sigma)$ if and only if

$$\begin{aligned} & \sum_{\omega \in E} \nu_\omega s_\omega v(q) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(p) > \sum_{\omega \in E} \nu_\omega s_\omega v(p) + \sum_{\omega' \in E^c} \nu_{\omega'} s_{\omega'} v(q) \\ \Leftrightarrow \quad & \sum_{\omega' \in E^c} s_{\omega'} \nu_{\omega'} [v(p) - v(q)] > \sum_{\omega \in E} s_\omega \nu_\omega [v(p) - v(q)] \\ \Leftrightarrow \quad & \sum_{\omega' \in E^c} s_{\omega'} \nu_{\omega'} > \sum_{\omega \in E} s_\omega \nu_\omega \end{aligned}$$

The final inequality holds because s satisfies property 2. Thus, $V^A(\sigma') > V^A(\sigma)$, as desired. □

D.2 Proofs for Section 6.2

Proof of Proposition 6

First, observe that if A is a (p, q)-bet and $\sigma^* \succ^A \sigma$, then DM1 and DM2 agree on the ranking of p and q. To see this, note that since $\sigma^* \succ^A \sigma$, DM1 cannot be indifferent between p and q. So, assume without loss of generality that $v(p) > v(q)$. Suppose toward a contradiction that $u(q) > u(p)$. It follows that $V^A(\sigma^*) = v(q)$, which is the lowest attainable payoff for DM1 in a (p, q)-bet. Hence, there can be no experiment such that $\sigma^* \succ^A \sigma$. Thus, we conclude that $u(p) \geq u(q)$.

Proof that (i) implies (ii). Suppose A is a (p, q)-bet and that $\sigma^* \succ^A \sigma$ for some σ. Assume without loss of generality that $v(p) > v(q)$. Then, by the argument above, we have $u(p) \geq u(q)$. By assumption, $\dot{v}(p) > \dot{v}(q)$; thus, by (i), we have $\dot{u}(p) \geq \dot{u}(q)$ as well. It follows that $\dot{V}^A(\sigma^*) = \dot{v}(p)$, which is the highest attainable payoff for D$\dot{\text{M}}$1 in a (p, q)-bet. Next, observe that $\dot{V}^A(\sigma) = \dot{v}(p)$ if and only if σ is equivalent to σ^* (that is, every $s \in \sigma$ is a scalar multiple of some $t \in \sigma^*$). But this would imply that $\sigma^* \sim^A \sigma$, a contradiction. Thus, $\dot{V}^A(\sigma) \neq \dot{v}(p)$, and therefore $\dot{V}^A(\sigma) < \dot{v}(p) = \dot{V}^A(\sigma^*)$. That is, $\sigma^* \mathrel{\dot{\succ}}^A \sigma$, as desired. □

Proof that (ii) implies (i). Suppose DM1 and DM2 agree on the ranking of p and q; without loss of generality, $v(p) > v(q)$ and $u(p) \geq u(q)$. Since $\dot{v}(p) > \dot{v}(q)$ (by assumption), it will suffice to show that $\dot{u}(p) \geq \dot{u}(q)$. So, let A be a (p, q)-bet. Since $v(p) > v(q)$ and $u(p) \geq u(q)$, there exists σ such that $\sigma^* \succ^A \sigma$ (for example, $\sigma = e$). By (ii), we have $\sigma^* \mathrel{\dot{\succ}}^A \sigma$. Since $\dot{v}(p) > \dot{v}(q)$, this forces $\dot{u}(p) \geq \dot{u}(q)$ by the argument above. □

Proof of Proposition 7

First, note that if A is a non-degenerate (p,q)-bet, then $v(p) \neq v(q)$. This implies that either $\sigma^* \succ^A e$ or $e \succ^A \sigma^*$. To see this, suppose without loss of generality that $v(p) > v(q)$. Then, either $u(p) \geq u(q)$ or $u(q) > u(p)$. If $u(p) \geq u(q)$, then $V^A(\sigma^*) = v(p)$ and $V^A(e) = \alpha v(p) + (1-\alpha)v(q)$ for some $\alpha \in (0,1)$; thus, $V^A(\sigma^*) > V^A(e)$. If instead $u(q) > u(p)$, then $V^A(\sigma^*) = v(q)$ and $V^A(e) = \alpha' v(p) + (1-\alpha')v(q)$ for some $\alpha' \in (0,1)$; thus, $V^A(e) > V^A(\sigma^*)$. Hence, either $\sigma^* \succ^A e$ or $e \succ^A \sigma^*$ whenever A is a non-degenerate bet.

A signal s belongs to the *EF-agreement region for DM* if, for all non-degenerate EF-bets A, the sets

$$\left\{ f \in A : \sum_\omega v(f_\omega)\nu^s_\omega \geq \sum_\omega v(g_\omega)\nu^s_\omega \ \forall g \in A \right\}$$

and

$$\left\{ f \in A : \sum_\omega u(f_\omega)\mu^s_\omega \geq \sum_\omega u(g_\omega)\mu^s_\omega \ \forall g \in A \right\}$$

are singletons and have nonempty intersection (that is, there exists a unique act $f \in A$ that maximizes expected utility under both (ν^s, v) and (μ^s, u)). Otherwise, s is in the *EF-disagreement region for DM*. A similar definition holds for the $\dot{E}F$-agreement/disagreement regions of $\dot{\text{D}}$M.

Lemma 34. *Suppose A is a non-degenerate EF-bet and that $\sigma \not\sim^A e$. If every $s \in \sigma$ belongs to the agreement region for DM, then σ is EF-extreme. Conversely, if σ is EF-extreme, then every $s \in \sigma$ belongs to the EF-agreement region.*

Proof. Let $A = \{pEq, pFq\}$ be non-degenerate. We will prove the lemma for the case $\sigma^* \succ^A e$, so that DM1 and DM2 agree on the ranking of p and q (the case $e \succ^A \sigma^*$ is symmetric). So, suppose without loss of generality that $v(p) > v(q)$ and $u(p) \geq u(q)$.

The first statement holds by the Blackwell information ordering. For the converse, suppose there exists $s \in \sigma$ such that s is in the disagreement region. If all signals of σ belong to the disagreement region, then $\succsim^A$ (locally) reverses the Blackwell ordering (and it is not difficult to construct a strict reversal), contradicting extremeness of σ.

Next, suppose some (but not all) signals of σ belong to the disagreement region. We will find signals $s, t \in \sigma$ such that s belongs to the disagreement region, t belongs to the agreement region, and $c^s(A) \neq c^t(A)$. Pick some $t' \in \sigma$ that belongs to the agreement region. Since $\sigma \not\sim^A e$, there is at least one $s' \in \sigma$ such that $c^{s'}(A) \neq c^{t'}(A)$. If every such s' is in the disagreement region, take $t = t'$ and $s = s'$ for such an s'. Otherwise, every $s' \in \sigma$ such that $c^{s'}(A) \neq c^{t'}(A)$ is in the agreement region. Hence, the set of all $s'' \in \sigma$ such that $c^{s''}(A) = c^{t'}(A)$ intersects the disagreement region (recall that σ contains at least one signal in the disagreement region). So, there exists $s \in \sigma$ such that $c^s(A) = c^{t'}(A)$ and s belongs to the disagreement region. Then take t to be any $s' \in \sigma$ such that $c^{s'}(A) \neq c^{t'}(A)$; our choice of s and t satisfies all requirements.

We will now show that $\succsim^A$ violates the Blackwell information ordering. Without loss of generality, assume $c^s(A) = pEq$ and $c^t(A) = pFq$. Since s is in the disagreement region, this implies $\sum_{\omega \in F} \nu_\omega s_\omega - \sum_{\omega \in E} \nu_\omega s_\omega > 0$ (that is, DM1 strictly prefers pFq at signal s). We may write $\sigma = [r^1, \ldots, r^K, t, s]$. Consider the garbling matrix M given by

$$M = \begin{bmatrix} I_K & 0 & \\ 0 & 1 & 0 \\ & 1-\alpha & \alpha \end{bmatrix}$$

where I_K denotes the $K \times K$ identity matrix. Then $\sigma' := \sigma M = [r_1, \ldots, r^K, t', s']$ where $t' = t + (1-\alpha)s$ and $s' = \alpha s$. Clearly, $c^{s'}(A) = c^s(A)$ and, for small enough $\alpha \in (0,1)$, $c^{t'}(A) = c^t(A)$. Thus, for $\alpha \in (0,1)$ sufficiently small,

$$\begin{aligned} \frac{V^A(\sigma') - V^A(\sigma)}{1-\alpha} &= -\left[\sum_{\omega \in E} \nu_\omega s_\omega v(p) + \sum_{\omega \in E^c} \nu_\omega s_\omega v(q)\right] + \left[\sum_{\omega \in F} \nu_\omega s_\omega v(p) + \sum_{\omega \in F^c} \nu_\omega s_\omega v(q)\right] \\ &= v(p)\left[\sum_{\omega \in F} \nu_\omega s_\omega - \sum_{\omega \in E} \nu_\omega s_\omega\right] - v(q)\left[\sum_{\omega \in E^c} \nu_\omega s_\omega - \sum_{\omega \in F^c} \nu_\omega s_\omega\right] \end{aligned}$$

Dividing both sides by $\sum_{\omega \in \Omega} \nu_\omega s_\omega$, it follows that $V^A(\sigma') - V^A(\sigma) > 0$ if and only if

$$v(p)\left[P(F|s) - P(E|s)\right] - v(q)\left[(1 - P(E|s)) - (1 - P(F|s))\right] > 0$$

where $P(E|s)$ and $P(F|s)$ are conditional probabilities of events E and F, respectively, given prior ν and signal s. Thus, $V^A(\sigma') - V^A(\sigma) > 0$ if and only if

$$(v(p) - v(q))\left[P(F|s) - P(E|s)\right] > 0$$

Since $v(p) > v(q)$, this is equivalent to $\sum_{\omega \in F} \nu_\omega s_\omega - \sum_{\omega \in E} \nu_\omega s_\omega > 0$. As demonstrated above, this condition is satisfied because s is in the disagreement region and $c^s(A) = pEq$. Thus, $V^A(\sigma') - V^A(\sigma) > 0$ and therefore $\sigma' \succ^A \sigma$. This violates the Blackwell information ordering because σ' is a garbling of σ. □

Proof that (i) ⇒ (ii). Suppose σ is EF-extreme. By Lemma 34, every $s \in \sigma$ is in the agreement region for DM1. By hypothesis, then, s is in the agreement region for $\dot{\text{D}}\text{M}$. Thus, by Lemma 34, σ is $E\dot{F}$-extreme. □

Proof that (ii) ⇒ (i). Suppose s is in the agreement region (for some E, F) for DM. Let σ be the experiment consisting of s and, for each ω such that $s_\omega < 1$, a signal (column) t such that $t_\omega = 1 - s_\omega$ and $t_{\omega'} = 0$ for all $\omega' \neq \omega$. Then each $s' \in \sigma$ is in the agreement region for DM (the additional signals t yield posteriors assigning probability 1 to a single state), so that by Lemma 34, σ is EF-extreme. By hypothesis, then, σ is also $\dot{E}F$-extreme. Therefore, by Lemma 34, s is in the agreement region for $\dot{\text{D}}$M. □

Proof that (ii) ⇔ (iii). Clearly, (iii) implies (ii). To see that (ii) implies (iii), let $A = \{pEq, pFq\}$ be non-degenerate and suppose $\sigma \in \mathcal{E}^*(A) \cap \dot{\mathcal{E}}^*(A)$ is EF-extreme and that $\sigma \sqsupseteq \sigma'$. Suppose $\sigma^* \mathrel{\dot{\succ}^A} e$, so that $\dot{\text{D}}$M1 and $\dot{\text{D}}$M2 agree on the ranking of p and q. Since every signal of σ belongs to the $\dot{E}F$-agreement region, we have $\sigma \mathrel{\dot{\succsim}^A} \sigma'$ (by the Blackwell ordering) even if every signal of σ' belongs to the agreement region. If one or more signals of σ' are in the disagreement region, then $\dot{V}^A(\sigma')$ is even lower (that is, $\dot{V}^A(\sigma')$ decreases as the set of signals of σ' in the disagreement region grows). Thus, in all cases, $\sigma \mathrel{\dot{\succsim}^A} \sigma'$. □

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